

# Operads and Motives in Deformation Quantization \*

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*Dedicated to the memory of Moshé Flato*

## Abstract

The algebraic world of associative algebras has many deep connections with the geometric world of two-dimensional surfaces. Recently D. Tamarkin discovered that the operad of chains of the little discs operad is formal, i.e. it is homotopy equivalent to its cohomology. From this fact and from Deligne's conjecture on Hochschild complexes follows almost immediately my formality result in deformation quantization. I review the situation as it looks now. Also I conjecture that the motivic Galois group acts on deformation quantizations, and speculate on possible relations of higher-dimensional algebras and of motives to quantum field theories.

## 1 Introduction

Deformation quantization was proposed about 22 years ago in the pioneering work of Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [BFFLS] as an alternative to the usual correspondence

$$\begin{array}{ccc} \text{classical systems} & \leftrightarrow & \text{quantum systems} \\ \text{symplectic manifolds} & \leftrightarrow & \text{Hilbert spaces} \end{array}$$

The idea is that algebras of observables in quantum mechanics are “close to” commutative algebras of functions on manifolds (phase spaces). In other words, quantum algebras of observables are *deformations* of commutative algebras.

In the first order in perturbation theory one obtains automatically a Poisson structure on phase space. Remind that a Poisson structure on a smooth manifold  $X$  is a bilinear

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operation  $\{\cdot, \cdot\}$  on  $C^\infty(X)$  satisfying the Jacobi identity and the Leibniz rule with respect to the usual product in  $C^\infty(X)$ . Two typical examples of Poisson manifolds are symplectic manifolds and dual spaces to Lie algebras. A star-product on a Poisson manifold  $X$  is an associative (but possibly non-commutative) product on  $C^\infty(X)$  depending formally on a parameter usually denoted by  $\hbar$  (the “Planck constant”). The product should have the form

$$f \star g = fg + \hbar \{f, g\} + \sum_{n \geq 2} \hbar^n B_n(f, g) \quad (1)$$

where  $(B_n)_{n \geq 1}$  are bidifferential operators  $C^\infty(X) \otimes C^\infty(X) \rightarrow C^\infty(X)$ . On the set of star-products acts an infinite-dimensional gauge group of linear transformations of the vector space  $C^\infty(X)$  depending formally on  $\hbar$ , of the form

$$f \mapsto f + \sum_{n \geq 1} \hbar^n D_n(f) \quad (2)$$

where the  $(D_n)_{n \geq 1}$  are differential operators on  $X$ .

About two years ago I proved (see [K1]) that for every Poisson manifold  $X$  there is a canonically defined gauge equivalence class of star-products on  $X$ . This result gave a complete answer to first basic questions in the program started by Moshé Flato and co-authors.

I obtained the existence of a canonical deformation quantization from a more general and stronger result, the formality theorem. The statement of this theorem is that in a suitably defined homotopy category of differential graded Lie algebras, two objects are equivalent. The first object is the Hochschild complex of the algebra of functions on the smooth manifold  $X$ , and the second object is a  $\mathbb{Z}$ -graded Lie superalgebra of polyvector fields on  $X$ . In the course of the proof I constructed an explicit isomorphism in the homotopy category of Lie algebras for the case  $X = \mathbb{R}^n$ . The terms in the formula for this isomorphism can be naturally identified with Feynman diagrams for certain two-dimensional quantum field theory with broken rotational symmetry (see [CF] for a detailed derivation of Feynman rules for this theory). The moral is that a kind of “string theory” is necessary for deformation quantization, which was originally associated with quantum mechanics! Moshé was extremely happy when I told him about my construction and we celebrated the ending of an old story with a bottle of champagne in his Paris apartment. Moshé was very enthusiastic about the new approach and expected new developments.

Later it became clear that not only there exists a canonical way to quantize, but that one can define a universal infinite-dimensional manifold parametrizing quantizations. There are several evidences that this universal manifold is a principal homogeneous space of the so-called Grothendieck-Teichmüller group, introduced by Drinfeld and Ihara. At the ICM98 Congress in Berlin, I gave a talk (which Moshé qualified as “wild”) on relations between deformations, motives and the Grothendieck-Teichmüller group. After the Congress I decided not to write notes of the talk because one month later a dramatic breakthrough in the area happened, which shed light on some parts of my talk and required further work; the present paper fills this gap.

Dmitry Tamarkin found a new short derivation of the formality theorem (for the case  $X = \mathbb{R}^n$ ) from a very general result concerning all associative algebras. Also from his result the (conjectured) relation between deformation quantizations and the Grothendieck-Teichmüller group seems to be much more transparent. The present paper is a result of my attempts to understand and generalize Tamarkin's results. I use all the time the language of operads and of homotopy theory of algebraic structures. For me it seems to be the first real application of operads to algebraic questions.

The main steps of the new proof of the formality theorem in deformation quantization are the following:

STEP 1) On the cohomological Hochschild complex of any associative algebra acts the operad of chains in the little discs operad (Deligne's conjecture). This result is purely topological/combinatorial.

STEP 2) The operad of chains of the little discs operad is formal, i.e. it is quasi-isomorphic to its cohomology. More precisely, this is true only in characteristic zero, e.g. over the field  $\mathbb{Q}$  of rational numbers. All the proofs of existence of a quasi-isomorphism use certain multi-dimensional integrals and give explicit formulas over real or over complex numbers.

STEP 3) From steps 1) and 2) follows that for any algebra  $A$  over a field of characteristic zero, its cohomological Hochschild complex  $C^*(A, A)$  and its Hochschild cohomology  $B := H^*(A, A)$  are algebras over the *same* operad (up to homotopy). Moreover, if one chooses an explicit homotopy of complexes between the Hochschild complex of  $A$  and the graded space  $B$  considered as a complex with zero differential, one obtains *two* different structures of a homotopy Gerstenhaber algebra on  $B$ . If these two structures are not equivalent, the first non-zero obstruction to the equivalence gives a non-zero element in the  $H^1(\text{Def}(B))$  of the deformation complex of the Gerstenhaber algebra  $B$ .

STEP 4) For the case  $A = \mathbb{R}[x_1, \dots, x_n]$ , the algebra  $B$  is the Gerstenhaber algebra of polynomial polyvector fields on  $\mathbb{R}^n$ . An easy calculation shows that  $H^1(\text{Def}(B))$  coincides with  $B^2$  (i.e. with the space of bivector fields on  $\mathbb{R}^n$ ). The explicit homotopy between  $B$  and  $C^*(A, A)$  can be made invariant under the group  $\text{Aff}(\mathbb{R}^n)$  of affine transformations of  $\mathbb{R}^n$ . There is no non-zero  $\text{Aff}(\mathbb{R}^n)$ -invariant bivector field on  $\mathbb{R}^n$ . The conclusion is that two structures of the homotopy Gerstenhaber algebra on  $B$ , mentioned in step 3), are equivalent. In particular, these two structures give equivalent structures of homotopy Lie algebra on  $B$ . This implies that  $C^*(A, A)$  is equivalent to  $B$  as a homotopy Lie algebra, which is the statement of the formality result in [K1].

Step 3) is the main discovery of D. Tamarkin. It is an absolutely fundamental new fact about *all* associative algebras. It implies in particular that the Hochschild complex is quasi-isomorphic to another natural complex with strictly associative and commutative cup-product. This other complex is hard to write down explicitly at the moment. Also, there are many different ways to identify operads as in step 2), even up to homotopy. All these choices form an infinite-dimensional algebraic manifold defined over the field of rational numbers  $\mathbb{Q}$ . On this manifold acts the Grothendieck-Teichmüller group. By the naturalness of the calculation in step 4), the same group acts on quantizations of Poisson structures on  $\mathbb{R}^n$ , and also on the set of gauge equivalence classes of star-products in  $\mathbb{R}^n$ .

Presumably, the action extends to general Poisson manifolds and to general star-products.

The paper is organized as follows:

Section 2: an introduction to operads, and to Deligne's conjecture concerning the Hochschild complex. I would like to apologize for certain vagueness in subsections 2.5-2.7.

Section 3: a proof of the result of Tamarkin on the formality of the chain operad of small discs operad, and a sketch of its application to my formality theorem in deformation quantization. In fact the idea of the proof presented here goes back to 1992-1993, but somehow at that time I missed the point.

Section 4: I describe in elementary terms a version of motives and of the motivic Galois group, and indicate its relations with homogeneous spaces appearing in various questions in deformation quantization.

Section 5: speculations about the possible rôle in Quantum Field Theories of things described in previous sections.

## 2 Deligne's conjecture and its generalization to higher dimensions

### 2.1 Operads and algebras

Here I remind the definitions of an operad and of an algebra over an operad (see also [GJ], [GK]). The language of operads is convenient for descriptions and constructions of various algebraic structures. It became quite popular in theoretical physics during the past few years because of the emergence of many new types of algebras related with quantum field theories.

**Definition 1** *An operad (of vector spaces) consists of the following:*

- 1) a collection of vector spaces  $P(n)$ ,  $n \geq 0$ ,
- 2) an action of the symmetric group  $S_n$  on  $P(n)$  for every  $n$ ,
- 3) an identity element  $\text{id}_P \in P(1)$ ,
- 4) compositions  $m_{(n_1, \dots, n_k)}$ :

$$P(k) \otimes (P(n_1) \otimes P(n_2) \otimes \dots \otimes P(n_k)) \longrightarrow P(n_1 + \dots + n_k) \quad (3)$$

for every  $k \geq 0$  and  $n_1, \dots, n_k \geq 0$  satisfying a natural list of axioms which will be clear from examples.

The simplest example of operad is given by  $P(n) := \text{Hom}(V^{\otimes n}, V)$  where  $V$  is a vector space. The action of the symmetric group and the identity element are obvious, and the compositions are defined by the substitutions

$$\begin{aligned} & (m_{(n_1, \dots, n_k)}(\phi \otimes (\psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_k)))(v_1 \otimes \dots \otimes v_{n_1+\dots+n_k}) \\ & := \phi(\psi_1(v_1 \otimes \dots \otimes v_{n_1}) \otimes \dots \otimes \psi_k(v_{n_1+\dots+n_{k-1}+1} \otimes \dots \otimes v_{n_1+\dots+n_k})) \end{aligned}$$

where  $\phi \in P(k) = \text{Hom}(V^{\otimes k}, V)$ ,  $\psi_i \in P(n_i) = \text{Hom}(V^{\otimes n_i}, V)$ ,  $i = 1, \dots, k$ .

This operad is called the endomorphism operad of a vector space. The axioms in the definition of operads express natural properties of this example. Namely, there should be an associativity law for multiple compositions, various compatibilities for actions of symmetric groups, and evident relations for compositions including the identity element.

Another important example of an operad is  $\text{Assoc}_1$ . The  $n$ -th component  $\text{Assoc}_1(n)$  for  $n \geq 0$  is defined as the collection of all universal (= functorial)  $n$ -linear operations  $A^{\otimes n} \rightarrow A$  defined on all associative algebras  $A$  with unit. The space  $\text{Assoc}_1(n)$  has dimension  $n!$ , and is spanned by the operations

$$a_1 \otimes a_2 \otimes \dots \otimes a_n \mapsto a_{\sigma(1)}a_{\sigma(2)} \dots a_{\sigma(n)}$$

where  $\sigma \in S_n$  is a permutation. The space  $\text{Assoc}_1(n)$  can be identified with the subspace of the free associative unital algebra in  $n$  generators consisting of expressions polylinear in each generator.

**Definition 2** An algebra over an operad  $P$  consist of a vector space  $A$  and a collection of polylinear maps  $f_n : P(n) \otimes A^{\otimes n} \rightarrow A$  for all  $n \geq 0$  satisfying the following list of axioms:

- 1) for any  $n \geq 0$  the map  $f_n$  is  $S_n$ -equivariant,
- 2) for any  $a \in A$  we have  $f_1(\text{id}_P \otimes a) = a$ ,
- 3) all compositions in  $P$  map to compositions of polylinear operations on  $A$ .

In other words, the structure of algebra over  $P$  on a vector space  $A$  is given by a homomorphism of operads from  $P$  to the endomorphism operad of  $A$ . Another name for algebras over  $P$  is  $P$ -algebras.

For example, an algebra over the operad  $\text{Assoc}_1$  is an associative unital algebra. If we replace the 1-dimensional space  $\text{Assoc}_1(0)$  by the zero space 0, we obtain an operad  $\text{Assoc}$  describing associative algebras possibly without unit. Analogously, there is an operad denoted  $\text{Lie}$ , such that  $\text{Lie}$ -algebras are Lie algebras. The dimension of the  $n$ -th component  $\text{Lie}(n)$  is  $(n - 1)!$  for  $n \geq 1$  and 0 for  $n = 0$ .

Let us warn the reader that not all algebraic structures correspond to operads. Two examples of classes of algebraic structures that cannot be cast in the language of operads are the class of fields and the class of Hopf algebras.

We conclude this section with an explicit description of free algebras in terms of operads.

**Theorem 1** Let  $P$  be an operad and  $V$  be a vector space. Then the free  $P$ -algebra  $\text{Free}_P(V)$  generated by  $V$  is naturally isomorphic as a vector space to

$$\bigoplus_{n \geq 0} (P(n) \otimes V^{\otimes n})_{S_n}$$

The subscript  $\cdot_{S_n}$  denotes the quotient space of coinvariants for the diagonal action of group  $S_n$ . The free algebra  $\text{Free}_P(V)$  is defined by the usual categorical adjunction property:

the set  $\text{Hom}_{P\text{-algebras}}(\text{Free}_P(V), A)$  (i.e. homomorphisms in the category of  $P$ -algebras) is naturally isomorphic to the set  $\text{Hom}_{\text{vector spaces}}(V, A)$  for any  $P$ -algebra  $A$ .

## 2.2 Topological operads, operads of complexes, etc.

In the definition of operads we can replace vector spaces by topological spaces, and the operation of tensor product by the usual (Cartesian) product.

**Definition 3** *A topological operad consists of the following:*

- 1) *a collection of topological spaces  $P(n)$ ,  $n \geq 0$ ,*
- 2) *a continuous action of the symmetric group  $S_n$  on  $P(n)$  for every  $n$ ,*
- 3) *an identity element  $\text{id}_P \in P(1)$ ,*
- 4) *compositions  $m_{(n_1, \dots, n_k)}$ :*

$$P(k) \times (P(n_1) \times P(n_2) \times \dots \times P(n_k)) \longrightarrow P(n_1 + \dots + n_k)$$

*which are continuous maps for every  $k \geq 0$  and  $n_1, \dots, n_k \geq 0$  satisfying a list of axioms analogous to the one in the definition of an operad of vector spaces.*

An analog of the endomorphism operad is the following one: for any  $n \geq 0$  the topological space  $P(n)$  is the space of continuous maps from  $X^n$  to  $X$ , where  $X$  is a given compact topological space.

In general, the definitions of an operad and of an algebra over an operad can be made in arbitrary symmetric monoidal category  $\mathcal{C}$  (i.e in a category endowed with the functor  $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ , the identity element  $1_{\mathcal{C}} \in \text{Objects}(\mathcal{C})$ , and various coherence isomorphisms for associativity, commutativity of  $\otimes$ , etc., see, for example, [ML]).

We shall here consider mainly operads in the symmetric monoidal category **Complexes** of  $\mathbb{Z}$ -graded complexes of abelian groups (or vector spaces over a given field). Often operads in the category of complexes are called *differential graded operads*, or simply *dg-operads*. Each component  $P(n)$  of an operad of complexes is a complex, i.e. a vector space decomposed into a direct sum  $P(n) = \bigoplus_{i \in \mathbb{Z}} P(n)^i$ , and endowed with a differential  $d : P(n)^i \longrightarrow P(n)^{i+1}$ ,  $d^2 = 0$ , of degree +1.

There is a natural way to construct an operad of complexes from a topological operad by using a version of the singular chain complex. Namely, for a topological space  $X$ , denote by **Chains**( $X$ ) the complex concentrated in nonpositive degrees, whose  $(-k)$ -th component for  $k = 0, 1, \dots$  consists of the formal finite additive combinations

$$\sum_{i=1}^N n_i \cdot f_i, \quad n_i \in \mathbb{Z}, \quad N \in \mathbb{Z}_{\geq 0}$$

of continuous maps  $f_i : [0, 1]^k \longrightarrow X$  (“singular cubes” in  $X$ ), modulo the following relations  
1)  $f \circ \sigma = \text{sign}(\sigma) f$  for any  $\sigma \in S_k$  acting on the standard cube  $[0, 1]^k$  by permutations of coordinates,

2)  $f' \circ pr_{k \rightarrow (k-1)} = 0$  where  $pr_{k \rightarrow (k-1)} : [0, 1]^k \longrightarrow [0, 1]^{k-1}$  is the projection onto first  $(k-1)$  coordinates, and  $f' : [0, 1]^{k-1} \longrightarrow X$  is a continuous map.

The boundary operator on cubical chains is defined in the usual way. The main advantage of cubical chains with respect to simplicial chains is that there is an external product map

$$\bigotimes_{i \in I} (\text{Chains}(X_i)) \longrightarrow \text{Chains}\left(\prod_{i \in I} X_i\right) \tag{4}$$

which is a *natural* homomorphism of complexes for any finite collection  $(X_i)_{i \in I}$  of topological spaces.

Now if  $P$  is a topological operad then the collection of complexes  $(\mathbf{Chains}(P(n)))_{n \geq 0}$  has a natural operad structure in the category of complexes of Abelian groups. The compositions in  $\mathbf{Chains}(P)$  are defined using the external tensor product of cubical chains.

Passing from complexes to their cohomology we obtain an operad  $H_*(P)$  of  $\mathbb{Z}$ -graded Abelian groups (= complexes with zero differential), the homology operad of  $P$ .

### 2.3 Operad of little discs

Let  $d \geq 1$  be an integer. Denote by  $G_d$  the  $(d+1)$ -dimensional Lie group acting on  $\mathbb{R}^d$  by affine transformations  $u \mapsto \lambda u + v$  where  $\lambda > 0$  is a real number and  $v \in \mathbb{R}^d$  is a vector. This group acts simply transitively on the space of closed discs in  $\mathbb{R}^d$  (in the usual Euclidean metric). The disc with center  $v$  and with radius  $\lambda$  is obtained from the standard disc

$$D_0 := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1^2 + \dots + x_d^2 \leq 1\}$$

by a transformation from  $G_d$  with parameters  $(\lambda, v)$ .

**Definition 4** *The little discs operad  $C_d$  is a topological operad with the following structure:*

- 1)  $C_d(0) = \emptyset$ ,
- 2)  $C_d(1) = \text{point} = \{\text{id}_{C_d}\}$ ,
- 3) for  $n \geq 2$  the space  $C_d(n)$  is the space of configurations of  $n$  disjoint discs  $(D_i)_{1 \leq i \leq n}$  inside the standard disc  $D_0$ .

The composition  $C_d(k) \times C_d(n_1) \times \dots \times C_d(n_k) \rightarrow C_d(n_1 + \dots + n_k)$  is obtained by applying elements from  $G_d$  associated with discs  $(D_i)_{1 \leq i \leq k}$  in the configuration in  $C_d(k)$  to configurations in all  $C_d(n_i)$ ,  $i = 1, \dots, k$  and putting the resulting configurations together. The action of the symmetric group  $S_n$  on  $C_d(n)$  is given by renumberations of indices of discs  $(D_i)_{1 \leq i \leq n}$ .

The operad  $C_d$  was introduced in 70-ies by Boardmann and Vogt, and by Peter May (see [BV], [M]) in order to describe homotopy types of  $d$ -fold loop spaces (i.e. spaces of continuous maps

$$\text{Maps}((S^d, \text{base point}), (X, x))$$

where  $X$  is a topological space with base point  $x$ ). It is the most important operad in homotopy theory. Strictly speaking, topologists use a slightly different model called the operad of little cubes, but the difference is not essential because homotopically there is no difference between cubes and discs.

The space  $C_d(n)$  is homotopy equivalent to the configuration space of  $n$  pairwise distinct points in  $\mathbb{R}^d$ .

$$\text{Conf}_n(\mathbb{R}^d) := (\mathbb{R}^d)^n \setminus \text{Diag} = \{(v_1, \dots, v_n) \in (\mathbb{R}^d)^n \mid v_i \neq v_j \text{ for any } i \neq j\}$$

There is an obvious continuous map  $C_d(n) \rightarrow \text{Conf}_n(\mathbb{R}^d)$  which associates a collection of disjoint discs with the collection of their centers. This map induces a homotopy equivalence because each fiber of this map is contractible.

The space  $\text{Conf}_2(\mathbb{R}^d)$  (and hence  $C_d(2)$ ) is homotopy equivalent to the  $(d-1)$ -dimensional sphere  $S^{d-1}$ . The homotopy equivalence is given by the map

$$(v_1, v_2) \mapsto \frac{v_1 - v_2}{|v_1 - v_2|} \in S^{d-1} \subset \mathbb{R}^d$$

## 2.4 Hochschild complex and Deligne's conjecture

In 1993 Pierre Deligne made a conjecture relating the little discs operad (in dimension  $d = 2$ ) and the cohomological Hochschild complex of an arbitrary associative algebra  $A$  (defined over any field  $k$ ). Namely, the Hochschild complex  $C^*(A, A)$  is concentrated in non-negative degrees and is defined as

$$C^n(A, A) := \text{Hom}_{\text{vector spaces}}(A^{\otimes n}, A), \quad n \geq 0 \quad (5)$$

and the differential in  $C^*(A, A)$  is given by the formula

$$\begin{aligned} (d\phi)(a_1 \otimes \dots \otimes a_{n+1}) &:= \\ a_1 \phi(a_2 \otimes \dots \otimes a_n) + \sum_{i=1}^n (-1)^i \phi(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) + \dots \\ &\dots + (-1)^{n+1} \phi(a_1 \otimes \dots \otimes a_n) a_{n+1} \end{aligned}$$

for any  $\phi \in C^n(A, A)$ .

The Hochschild complex plays a fundamental rôle in the deformation theory of an associative algebra  $A$ . There are two basic operations on  $C^* := C^*(A, A)$ , the cup product  $\cup : C^k \otimes C^l \rightarrow C^{k+l}$  and the Gerstenhaber bracket  $[ , ] : C^k \otimes C^l \rightarrow C^{k+l-1}$ . The formulas for these operations are the following (here  $\phi \in C^k$ ,  $\psi \in C^l$ ):

$$\begin{aligned} (\phi \cup \psi)(a_1 \otimes \dots \otimes a_{k+l}) &:= (-1)^{kl} \phi(a_1 \otimes \dots \otimes a_k) \cdot \psi(a_{k+1} \otimes \dots \otimes a_{k+l}) \\ [\phi, \psi] &:= \phi \circ \psi - (-1)^{(k-1)(l-1)} \psi \circ \phi \quad \text{where} \\ (\phi \circ \psi)(a_1 \otimes \dots \otimes a_{k+l-1}) &:= \\ &\sum_{i=1}^{k-1} (-1)^{i(l-1)} \phi(a_1 \otimes \dots \otimes a_i \otimes \psi(a_{i+1} \otimes \dots \otimes a_{i+l}) \otimes \dots \otimes a_{k+l-1}) \end{aligned}$$

The Gerstenhaber bracket gives (after a shift of the  $\mathbb{Z}$ -grading by 1) the structure of differential graded Lie algebra on the Hochschild complex. The cup product has also a remarkable property: it is *not* graded commutative (it is associative only), but the induced operation on the cohomology *is* graded commutative. Moreover, the Gerstenhaber bracket

induces an operation on  $H^*(A, A)$  which satisfies the Leibniz rule with respect to the cup product:

$$[\phi, \psi_1 \cup \psi_2] = [\phi, \psi_1] \cup \psi_2 + (-1)^{(k-1)(l_1-1)} \psi_1 \cup [\phi, \psi_2], \quad \phi \in C^k, \psi_i \in C^{l_i}$$

although this identity does not hold on the level of cochains. The cohomology space  $H^*(A, A)$  carries the structure of the Gerstenhaber algebra, i.e. it is a graded vector space endowed with a Lie bracket of degree  $(-1)$  (satisfying the skew symmetry condition and the Jacobi identity with appropriate signs), and with a graded commutative associative product of degree  $0$ , satisfying the graded Leibniz rule with respect to the bracket.

It was observed by several people (F. Cohen in [C], P. Deligne,...) that the  $\mathbb{Z}$ -graded operad **Gerst** describing Gerstenhaber algebras has a very beautiful topological meaning. Namely, it is naturally equivalent to the homology operad of the topological operad  $C_2$ . The space of binary operations  $C_2(2)$  is homotopy equivalent to the circle  $S^1$ . The Gerstenhaber bracket corresponds to the generator of  $H_1(S^1) \simeq \mathbb{Z}$  and the cup product corresponds to the generator of  $H_0(S^1) \simeq \mathbb{Z}$ .

**Conjecture 1 (P. Deligne)** *There exists a natural action of the operad  $\text{Chains}(C_2)$  on the Hochschild complex  $C^*(A, A)$  for arbitrary associative algebra  $A$ .*

The story of this conjecture is quite dramatic. In 1994 E. Getzler and J. Jones posted on the e-print server a preprint [GJ] in which the proof of the Deligne conjecture was contained. Essentially at the same time M. Gerstenhaber and A. Voronov in [GV] made analogous claims. The result was considered as well established and was actively used later. But in the spring of 1998 D. Tamarkin observed that there was a serious flaw in both preprints. The cell decomposition used there turned out to be not compatible with the operad structure; the first example of wrong behavior appears for operations with 6 arguments.

I think that I have now a complete proof of the Deligne conjecture (and its generalization, see the next section). A combination of two results of Tamarkin (see [T1] and [T2]) also implies Deligne conjecture. Sasha Voronov says that he corrected the problem in his approach, and there is also an announcement [MS] by J. McClure and J. Smith with the same result.

Unfortunately, all “proofs” and announcements of proof are still too complicated to be put here. We need a really short and convincing argument for this very fundamental fact about Hochschild complexes. It seems that the simplicity of the Hochschild complex is deceiving.

## 2.5 Higher-dimensional generalization of Deligne’s conjecture

I propose here an “explanation” of Deligne’s conjecture and its natural generalization.

The operad **Assoc** has a topological origin, it is the homology operad of the little intervals (i.e. 1-dimensional discs) operad. Namely, for  $n \geq 1$ , the spaces  $C_1(n)$  have  $n!$

connected components corresponding to permutations  $\sigma \in S_n$ , and each of the components is contractible.

We can go still one step down, defining the operad  $C_0$  as a really trivial operad:

$$C_0(n) = \emptyset \text{ for } n \neq 1, \quad C_0(1) = \text{point} = \{\text{id}_{C_0}\}$$

This definition is natural because one cannot put  $\geq 2$  disjoint zero-dimensional discs inside one zero-dimensional disc (= point).

**Definition 5** For  $d \geq 0$ , a  $d$ -algebra is an algebra over the operad  $\text{Chains}(C_d)$  in the category of complexes.

For  $d > 0$  this notion was introduced by Getzler and Jones. By definition, a 0-algebra is just a complex. The notion of 1-algebra is very close to the notion of an associative algebra. There is a well-known companion of associative algebras, a class of so called  $A_\infty$ -algebras. One can show that any 1-algebra carries a natural structure of  $A_\infty$ -algebra, and from the points of view of homotopy theory and of deformation theory, there is no difference between associative algebras,  $A_\infty$ -algebras and 1-algebras (see 2.6 and 3.1). In particular, one can introduce the cohomological Hochschild complex for  $A_\infty$ -algebras and 1-algebras. This Hochschild complex carries a natural structure of a differential graded Lie algebra.

An  $A_\infty$ -version of Deligne's conjecture says that this Hochschild complex carries naturally a structure of 2-algebra, extending the structure of differential graded Lie algebra. It has a baby version in dimension  $(0+1)$ : if  $A$  is a vector space (i.e. 0-algebra concentrated in degree 0) then the Lie algebra of the group of affine transformations

$$\text{Lie}(\text{Aff}(A)) = \text{End}(A) \oplus A$$

has also a natural structure of an associative algebra, in particular it is a 1-algebra. The product in  $\text{End}(A) \oplus A$  is given by the formula

$$(\phi_1, a_1) \times (\phi_2, a_2) := (\phi_1 \phi_2, \phi_1(a_2)) .$$

The space  $\text{End}(A) \oplus A$  plays the rôle of the Hochschild complex in the case  $d = 0$  (see 2.7).

The definition of a  $d$ -algebra given above seems to be too complicated for doing concrete calculations. At the end of section 3.2 we describe much smaller operads which are quasi-isomorphic to the chain operads of little discs operads. The reader will see that in dimension  $d \geq 2$  the situation is very simple.

Now we introduce the notion of *action* of a  $(d+1)$ -algebra on a  $d$ -algebra. It is convenient to formulate it using so-called colored operads. Instead of defining exactly what a colored operad is, we give one typical example: there is a colored operad with two colors such that algebras over this operad are pairs  $(\mathfrak{g}, A)$  where  $\mathfrak{g}$  is a Lie algebra and  $A$  is an associative algebra on which  $\mathfrak{g}$  acts by derivations.

Let us fix a dimension  $d \geq 0$ . Denote by  $\sigma : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  the reflection

$$(x_1, \dots, x_{d+1}) \mapsto (x_1, \dots, x_d, -x_{d+1})$$

at the coordinate hyperplane, and by  $\mathcal{H}_+$  the upper-half space

$$\{(x_1, \dots, x_{d+1}) \mid x_{d+1} > 0\}$$

**Definition 6** For any pair of non-negative integers  $(n, m)$  we define a topological space  $SC_d(n, m)$  as

- 1) the empty space  $\emptyset$  if  $n = m = 0$ ,
- 2) the one-point space if  $n = 0$  and  $m = 1$ ,
- 3) in the case  $n \geq 1$  or  $m \geq 2$ , the space of configurations of  $m + 2n$  disjoint discs  $(D_1, \dots, D_{m+2n})$  inside the standard disc  $D_0 \subset \mathbb{R}^{d+1}$  such that  $\sigma(D_i) = D_i$  for  $i \leq m$ ,  $\sigma(D_i) = D_{i+n}$  for  $m + 1 \leq i \leq m + n$  and such that all discs  $D_{m+1}, \dots, D_{m+n}$  are in the upper half space  $\mathcal{H}_+$ .

The reader should think about points of  $SC_d(n, m)$  as about configurations of  $m$  disjoint semidisks  $(D_1 \cap \mathcal{H}_+, \dots, D_m \cap \mathcal{H}_+)$  and of  $n$  disks  $(D_{m+1}, \dots, D_{m+n})$  in the standard semidisk  $D_0 \cap \mathcal{H}_+$ . The letters “SC” stand for “Swiss Cheese” [V]. Notice that the spaces  $SC_d(0, m)$  are naturally isomorphic to  $C_d(m)$  for all  $m$ . One can define composition maps analogously to the case of the operad  $C_d$ :

$$\begin{aligned} SC_d(n, m) \times (C_{d+1}(k_1) \times \dots \times C_{d+1}(k_n)) \times (SC_d(a_1, b_1) \times \dots \times SC_d(a_m, b_m)) \\ \longrightarrow SC_d(k_1 + \dots + k_n + a_1 + \dots + a_m, b_1 + \dots + b_m) \end{aligned}$$

**Definition 7 (A .Voronov [V])** The colored operad  $SC_d$  has two colors and consists of collections of spaces

$$(SC_d(n, m))_{n, m \geq 0}, \quad (C_{d+1}(n))_{n \geq 0},$$

and appropriate actions of symmetric groups, identity elements, and of all composition maps.

As before, we can pass from a colored operad of topological spaces to a colored operad of complexes using the functor **Chains**.

**Definition 8** An action of a  $(d + 1)$ -algebra  $B$  on a  $d$ -algebra  $A$  is, on the pair  $(B, A)$ , a structure of algebra over the colored operad **Chains**( $CS_d$ ), compatible with the structures of algebras on  $A$  and on  $B$ .

The generalized Deligne conjecture says that for every  $d$ -algebra  $A$  there exists a universal (in an appropriate sense, up to homotopy, see 2.6)  $(d + 1)$ -algebra acting on  $A$ . I think that I have a proof of this conjecture, so I am making the following

**Claim 1** Let  $A$  be a  $d$ -algebra for  $d \geq 0$ . Consider the homotopy category of pairs  $(B, \rho)$  where  $B$  is a  $(d + 1)$ -algebra and  $\rho$  is an action of  $B$  on  $A$ . In this category there exists a final object.

Using this claim one can give the

**Definition 9** *For a d-algebra A the (generalized) Hochschild complex  $\text{Hoch}(A)$  is defined as the universal  $(d+1)$ -algebra acting on A.*

The universality of  $\text{Hoch}(A)$  means that in the sense of homotopy theory of algebras, an action of  $(d+1)$ -algebra  $B$  on  $A$  is the same as a homomorphism of  $(d+1)$ -algebras  $B \rightarrow \text{Hoch}(A)$ .

In the next subsection we review homotopy and deformation theory of algebras over operads, and in the following one we describe an “explicit” model for  $\text{Hoch}(A)$ .

## 2.6 Homotopy theory and deformation theory

Let  $P$  be an operad of complexes, and  $f : A \rightarrow B$  be a morphism of two  $P$ -algebras.

**Definition 10** *f is a quasi-isomorphism iff it induces an isomorphism of the cohomology groups of A and B considered just as complexes.*

Two algebras  $A$  and  $B$  are called *homotopy equivalent* iff there exists a chain of quasi-isomorphisms

$$A = A_1 \rightarrow A_2 \leftarrow A_3 \rightarrow \dots \leftarrow A_{2k+1} = B \quad (6)$$

One can define a new structure of category on the collection of  $P$ -algebras in which quasi-isomorphic algebras become equivalent. There are several ways to do it, using either Quillen’s machinery of homotopical algebra (see [Q]), or using a free resolution of the operad  $P$ , or some simplicial constructions, etc. For example, in the category of differential graded Lie algebras, morphisms in the homotopy category are so called  $L_\infty$ -morphisms (see e.g. [K1]), modulo a suitably defined equivalence relation (a homotopy between morphisms). I shall not give here any precise definition of the homotopy category in general, just say that morphisms in the homotopy category of  $P$ -algebras are connected components of certain topological spaces, exactly as in the usual framework of homotopy theory (i.e. in the category of topological spaces).

In the case when the operad  $P$  satisfies some technical conditions, one can transfer the structure of a  $P$ -algebra by quasi-isomorphisms of complexes. In particular, one can make the construction described in the following lemma:

**Lemma 1** *Let  $P$  be an operad of complexes, such that if we consider  $P$  as an operad just of  $\mathbb{Z}$ -graded vector spaces, it is free and generated by operations in  $\geq 2$  arguments. Let  $A$  be an algebra over  $P$ , and let us choose a splitting of  $A$  considered as a  $\mathbb{Z}$ -graded space into the direct sum*

$$A = H^*(A) \oplus V \oplus V[-1], \quad (V[-1])^k := V^{k-1}$$

*endowed with a differential of the form  $d(a \oplus b \oplus c) = 0 \oplus 0 \oplus b[-1]$ . Then there is a canonical structure of a  $P$ -algebra on the cohomology space  $H^*(A)$ , and this algebra is homotopy equivalent to  $A$ .*

We shall use it later (in 3.5) in combination with the fact that the operad  $\mathbb{Q} \otimes \text{Chains}(C_d)$  is free as an operad of  $\mathbb{Z}$ -graded vector spaces over  $\mathbb{Q}$ . This is evident because the action of  $S_n$  on  $C_d(n)$  is free and the composition morphisms in  $C_d$  are embeddings.

One can associate with any operad  $P$  of complexes and, with any  $P$ -algebra  $A$ , some differential graded Lie algebra (or more generally, a  $L_\infty$ -algebra, see 3.1)  $\text{Def}(A)$ . This Lie algebra is defined canonically up to a quasi-isomorphism (the same as up to a homotopy). It controls the deformations of  $P$ -algebra structure on  $A$ . There are several equivalent constructions of  $\text{Def}(A)$  using either resolutions of  $A$  or resolutions of the operad  $P$ . Morally,  $\text{Def}(A)$  is the Lie algebra of derivations in homotopy sense of  $A$ . For example, if  $P$  is an operad with zero differential then  $\text{Def}(A)$  is quasi-isomorphic to the differential graded Lie algebra of derivations of  $\tilde{A}$  where  $\tilde{A}$  is any free resolution of  $A$ .

The differential graded Lie algebras  $\text{Def}(A_1)$  and  $\text{Def}(A_2)$  are quasi-isomorphic for homotopy equivalent  $P$ -algebras  $A_1$  and  $A_2$ .

## 2.7 Hochschild complexes and deformation theory

First of all, if  $A$  a  $d$ -algebra then the shifted complex  $A[d - 1]$ ,

$$(A[d - 1])^k := A^{(d-1)+k}$$

carries a natural structure of  $L_\infty$ -algebra. It comes from a homomorphism of operads in homotopy sense from the twisted by  $[d - 1]$  operad  $\text{Chains}(C_d)$  to the operad  $\text{Lie}$ . In order to construct such a homomorphism one can use fundamental chains of all components of the Fulton-MacPherson operad (see 3.3.1), or deduce the existence of a homomorphism from the results in 3.2.

Moreover,  $A[d - 1]$  maps as homotopy Lie algebra to  $\text{Def}(A)$ , i.e.  $A[d - 1]$  maps to “inner derivations” of  $A$ . These inner derivations form a Lie ideal in  $\text{Def}(A)$  in homotopy sense.

**Claim 2** *The quotient homotopy Lie algebra  $\text{Def}(A)/A[d - 1]$  is naturally quasi-isomorphic to  $\text{Hoch}(A)[d]$ .*

In the case when  $d = 0$  and the complex  $A$  is concentrated in degree 0, the Lie algebra  $\text{Def}(A)$  is  $\text{End}(A)$ , i.e. the Lie algebra of linear transformations in the vector space  $A$ .

**Lemma 2** *The Hochschild complex of 0-algebra  $A$  is  $A \oplus \text{End}(A)$  (placed in degree 0).*

Here follows a sketch of the proof. First of all, the colored operad  $SC_0$  is quasi-isomorphic to its zero-homology operad  $H_0(SC_0)$  because all connected components of spaces  $(SC_0(n, m))_{n, m \geq 0}$  and of  $(C_1(n))_{n \geq 0}$  are contractible. By general philosophy (see 3.1) this implies that we can replace  $SC_0$  by  $H_0(SC_0)$  in the definition of the Hochschild complex given in 2.5. The  $H_0$ -version of a 1-algebra is an associative non-unital algebra, and the  $H_0$ -version of an action is the following:

1) an action of an associative non-unital algebra  $B$  on vector space  $A$  (it comes from the generator of  $\mathbb{Z} = H_0(SC_0(1, 1))$ ),

2) a homomorphism from  $B$  to  $A$  of  $B$ -modules (coming from the generator of  $\mathbb{Z} = H_0(SC_0(1, 0))$ ).

It is easy to see that to define an action as above is the same as to define a homomorphism of non-unital associative algebras from  $B$  to  $\text{End}(A) \oplus A$ . Thus, the Hochschild complex is (up to homotopy) equal to  $\text{End}(A) \oplus A$ .

Let us continue the explanation of the Claim 2 above for the case  $d = 0$ . As (homotopy) Lie algebra  $\text{Hoch}(A)$  coincides with the Lie algebra of affine transformations on  $A$ . The homomorphism  $\text{Def}(A) \rightarrow \text{Hoch}(A)$  is a *monomorphism*, but in homotopy category every morphism of Lie algebras can be replaced by an epimorphism! The Abelian Lie superalgebra  $A[-1]$  is the “kernel” of this morphism. Let us show explicitly how all this works. The Lie algebra  $\text{Def}(A) = (A)$  is quasi-isomorphic to the following differential graded Lie algebra  $\mathfrak{g}$ : as  $\mathbb{Z}$ -graded space it is

$$\text{End}(A) \oplus A \oplus A[-1],$$

i.e. the graded components of  $\mathfrak{g}$  are  $\mathfrak{g}^0 = \text{End}(A) \oplus A$ ,  $\mathfrak{g}^1 = A$ ,  $\mathfrak{g}^{>0,1} = 0$ . The nontrivial components of the Lie bracket on  $\mathfrak{g}$  are the usual bracket on  $\text{End}(A)$  and the action of  $\text{End}(A)$  on  $A$  and on  $A[-1]$ . The only nontrivial component of the differential on  $\mathfrak{g}$  is the shifted by [1] identity map from  $A$  to  $A[-1]$ . The evident homomorphism

$$\mathfrak{g} \rightarrow \text{End}(A)$$

is a homomorphism of differential graded Lie algebras, and also a quasi-isomorphism. There is a short exact sequence of dg-Lie algebras

$$0 \rightarrow A[-1] \rightarrow \mathfrak{g} \rightarrow \text{End}(A) \oplus A \rightarrow 0$$

In the case  $d = 1$  an analogous thing happens. The deformation complex of an associative algebra  $A$  is the following *subcomplex* of the shifted by [1] Hochschild complex:

$$\text{Def}(A)^n := \text{Hom}_{\text{vector spaces}}(A^{\otimes(n+1)}, A) \text{ for } n \geq 0; \quad \text{Def}^{<0}(A) := 0$$

The deformation complex is quasi-isomorphic to an  $L_\infty$ -algebra  $\mathfrak{g}$  which as  $\mathbb{Z}$ -graded vector space is

$$\text{Def}(A) \oplus A \oplus A[1].$$

The Hochschild complex of  $A$  is a *quotient* complex of  $\mathfrak{g}$  by the homotopy Lie ideal  $A$ .

### 3 Formality of operads of chains, application to deformation quantization

#### 3.1 Quasi-isomorphisms of operads

Operads themselves are algebras over a certain colored operad, Operads. This is quite obvious because an operad is just a collection of vector spaces and polylinear maps between these spaces satisfying some identities. If we work in characteristic zero, it is convenient

to associate colors with all Young diagrams, i.e. with all irreducible representations of all finite symmetric groups  $S_n$ ,  $n \geq 0$ .

Analogously to the case of algebras, we can speak about quasi-isomorphisms of operads of complexes.

**Definition 11** *A morphism  $f : P_1 \rightarrow P_2$  between two operads of complexes is called a quasi-isomorphism iff the maps of complexes  $f(n) : P_1(n) \rightarrow P_2(n)$  induce isomorphisms of cohomology groups for all  $n$ .*

Homotopy categories and deformation theories of algebras over quasi-isomorphic operads are equivalent. A typical example: the operad **Lie** is quasi-isomorphic to the operad  $L_\infty$  describing  $L_\infty$ -algebras. Remind that a  $L_\infty$ -algebra  $V$  is a complex of vector spaces endowed with a coderivation  $d_C$  of degree (+1) of the cofree cocommutative coassociative  $\mathbb{Z}$ -graded coalgebra without counit cogenerated by  $V[1]$ :

$$C := \bigoplus_{n \geq 1} ((V[1])^{\otimes n})_{S_n}$$

such that the component of  $d_C$  mapping  $V$  to  $V$  coincides (up to a shift) with the differential  $d_V$  of  $V$ . Analogously, the operad **Assoc** is quasi-isomorphic to the operad **Chains**( $C_1$ ), and also to the operad  $A_\infty$  responsible for  $A_\infty$ -algebras.

### 3.2 Formality of chain operads

**Theorem 2** *The operad **Chains**( $C_d$ )  $\otimes \mathbb{R}$  of complexes of real vector spaces is quasi-isomorphic to its cohomology operad endowed with zero differential.*

In general, differential graded algebras quasi-isomorphic to their cohomology endowed with zero differential, are called formal. The classical example is the de Rham complex of a compact Kähler manifold. The result of Deligne-Griffiths-Morgan-Sullivan (see [DGMS]) is that this algebra is formal as differential graded commutative associative algebra. Our theorem says that **Chains**( $C_d$ )  $\otimes \mathbb{R}$  is formal as an algebra over the colored operad **Operads**.

The story of this theorem is truly complicated. It was stated in the preprint of Getzler and Jones originally for the cases  $d = 1$  and  $d = 2$ . The case  $d = 1$  is trivial (as well as  $d = 0$ ). The authors referred to me for the claim that **Chains**( $C_d$ )  $\otimes \mathbb{R}$  is not formal for  $d \geq 3$ . Later it became clear that the proof in the Getzler-Jones preprint is not correct. Personally, I thought for several years that even the fact is not true, and made several times calculations demonstrating the non-formality of **Chains**( $C_2$ )  $\otimes \mathbb{R}$ . The story began to stir again when D. Tamarkin posted on the net a new proof of the formality theorem in deformation quantization. The basic intermediate result in Tamarkin's approach was the formality of **Chains**( $C_2$ )  $\otimes \mathbb{R}$  for which he claimed a new (with respect to [GJ]) proof. Unfortunately, there were several mistakes in Tamarkin's proof at that time as well, until a new corrected version was posted (see [T1]) which did not use Deligne's conjecture directly. In [T1] is proven that  $H_*(C_2)$  acts on the Hochschild complex of any algebra  $A$ .

Also Tamarkin found a proof that  $\text{Chains}(C_2) \otimes \mathbb{C}$  is quasi-isomorphic to  $H_*(C_2) \otimes \mathbb{C}$  (see [T2]). Deligne's conjecture follows from the combination of two results of Tamarkin, but unfortunately not in a purely topological/combinatorial way. In the meantime I found a flaw in my calculations which "show" non-formality of  $\text{Chains}(C_2) \otimes \mathbb{R}$ , and found a new proof of the formality of  $\text{Chains}(C_d) \otimes \mathbb{R}$  valid for all  $d$ . The quasi-isomorphism which I constructed differs essentially from the one constructed by Tamarkin in [T2]. It seems that it gives really different deformation quantizations of Poisson manifolds. The question of differences between quantizations is addressed in the third part of this article.

I finish this subsection by a description of the homology operads of topological operads  $C_d$ .

**Theorem 3** *An algebra over  $H_*(C_d)$  is*

- 1) *a complex in the case  $d = 0$ ,*
- 2) *a differential graded associative algebra in the case  $d = 1$ ,*
- 3) *a differential graded "twisted" Poisson algebra with the commutative associative product of degree 0 and with the Lie bracket of degree  $(1 - d)$  in the case of odd  $d \geq 3$ ,*
- 4) *a differential graded "twisted" Gerstenhaber algebra with the commutative associative product of degree 0 and with the Lie bracket of degree  $(1 - d)$  in the case of even  $d \geq 2$ .*

*In cases 3) and 4) the Lie bracket satisfies the Leibniz rule with respect to the product.*

The "twisting" in cases 3) and 4) means just that the commutator has the usual  $\mathbb{Z}/2\mathbb{Z}$ -grading, but a *different*  $\mathbb{Z}$ -grading. The total rank of the homology group  $H_*(C_d(n))$  is  $n!$  for all  $d \geq 1$ . The rank of the top degree homology group of  $C_d(n)$  is  $(n - 1)!$  for any  $d \geq 2$  and  $n \geq 1$ . As a representation of  $S_n$  this homology group  $H_{(n-1)(d-1)}(C_d(n))$  is isomorphic (up to tensoring by the sign representation of  $S_n$  for even  $n$ ) to  $\text{Lie}(n)$ , the  $n$ -th space of the Lie operad.

The conclusion from two theorems in this section is that  $d$ -algebras for  $d \geq 2$  are essentially the same as twisted Poisson or Gerstenhaber algebras (depending on the parity of  $d$ ).

### 3.3 Sketch of the proof of the formality of chain operads

The proof presented here is quite technical and it is not really essential for the rest of this paper. The reader can skip it and go directly to section 3.4.

#### 3.3.1 Fulton-MacPherson compactification

Let us fix dimension  $d \geq 1$ . There is a modification  $FM_d$  of the topological operad  $C_d$  which is more convenient to work with. The idea to consider operad  $FM_d$  was proposed by several people, in particular by Getzler and Jones in [GJ]. The letters "FM" stand for Fulton and MacPherson, who introduced closely related constructions in the realm of algebraic geometry (see [FM]). I shall describe now the operad  $FM_d$ .

For  $n \geq 2$  denote by  $\tilde{C}_d(n)$  the quotient space of the configuration space of  $n$  points in  $\mathbb{R}^d$

$$\text{Conf}_n(\mathbb{R}^d) := \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n \mid x_i \neq x_j \text{ for any } i \neq j\}$$

modulo the action of the group  $G_d = \{x \mapsto \lambda x + v \mid \lambda \in \mathbb{R}_{>0}, v \in \mathbb{R}^d\}$ . The space  $\tilde{C}_d(n)$  is a smooth manifold of dimension  $(nd - d - 1)$ . For  $n = 2$ , the space  $\tilde{C}_d(n)$  coincides with the  $(d - 1)$ -dimensional sphere  $S^{d-1}$ . There is an obvious free action of  $S_n$  on  $\tilde{C}_d(n)$ . We define the spaces  $\tilde{C}_d(0)$  and  $\tilde{C}_d(1)$  to be empty. The collection of spaces  $\tilde{C}_d(n)$  does not form an operad because there is no identity element, and compositions are not defined.

The components of the operad  $FM_d$  are

- 1)  $FM_d(0) := \emptyset$ ,
- 2)  $FM_d(1) = \text{point}$ ,
- 3)  $FM_d(2) = \tilde{C}_d(2) = S^{d-1}$ ,
- 4) for  $n \geq 3$  the space  $FM_d(n)$  is a manifold with corners, its interior is  $C'_d(n)$ , and all boundary strata are certain products of copies of  $\tilde{C}_d(n')$  for  $n' < n$ .

A manifold with corners looks locally as a product of manifolds with boundary. Any manifold with corners is automatically a *topological* manifold, although its *smooth* structure is not the one of a differentiable manifold with boundary.

Intuitively, if a configuration of  $n$  points in  $\mathbb{R}^d$  moves in such a way that several points (or groups of points) become at the limit close to each other, we use a microscope with a very large magnification (apply a large element of the group  $G_d$ ) in order to see in details the shape of configurations of points in clusters. One of the rigorous definitions of  $FM_d(n)$  is the following:

**Definition 12** For  $n \geq 2$ , the manifold with corners  $FM_d(n)$  is the closure of the image of  $\tilde{C}_d(n)$  in the compact manifold  $(S^{d-1})^{n(n-1)/2}$  under the map

$$G_d \cdot (x_1, \dots, x_n) \mapsto \left( \frac{x_j - x_i}{|x_j - x_i|} \right)_{1 \leq i < j \leq n}$$

Set-theoretically, the operad  $FM_d$  is the same as the free operad generated by the collection of sets  $(\tilde{C}_d(n))_{n \geq 0}$  endowed with  $S_n$ -actions as above.

It is possible to define another topological operad  $FM'_d$ , and two homomorphisms of operads

$$f_1 : C_d \longrightarrow FM'_d, \quad f_2 : FM_d \longrightarrow FM'_d$$

such that for all  $n \geq 0$  maps

$$f_1(n) : C_d(n) \longrightarrow FM'_d(n), \quad f_2(n) : FM_d(n) \longrightarrow FM'_d(n)$$

are homotopy equivalences. In a sense, the spaces  $FM'_d(n)$  parametrize configurations of small disks in the standard disk, together with a class of “degenerate” configurations in which some (or all) disks are infinitely small. We leave as an exercise to the reader to give a complete definition of  $FM'_d$ .

Applying the functor **Chains** we get two quasi-isomorphisms of operads of complexes

$$\text{Chains}(C_d) \longrightarrow \text{Chains}(FM'_d)$$

$$\text{Chains}(FM'_d) \longleftarrow \text{Chains}(FM_d)$$

### 3.3.2 A chain of quasi-isomorphisms

For any  $d \geq 2$  I shall define several operads of complexes and construct a chain of quasi-isomorphisms between them. The total diagram is the following:

$$\begin{aligned} \text{Chains}(FM_d) &\longleftarrow \text{SemiAlgChains}(FM_d) \\ \text{SemiAlgChains}(FM_d) \otimes \mathbb{R} &\longrightarrow \text{Graphs}_d \hat{\otimes} \mathbb{R} \\ \text{Graphs}_d &\longleftarrow \text{Forests}_d \longleftarrow H_*(\text{Forests}) H_*(C_d) \end{aligned} \tag{7}$$

I shall now explain the first line. It is really technical, and is introduced only to circumvent some difficulties with integrals which appear in the second line (see the next section).

The operad **SemiAlgChains**( $FM_d$ ) is the suboperad of **Chains**( $FM_d$ ) consisting of combinations of maps  $[0, 1]^k \longrightarrow FM_d(n)$  whose graphs are real semi-algebraic sets. This is well defined because the space  $FM_d(n)$  can be described in terms of algebraic equations and inequalities. It is a closed semi-algebraic subset in the product of  $\frac{1}{2}n(n - 1)$  copies of  $S^{d-1}$ . The natural inclusion of semi-algebraic chains into all continuous chains is a quasi-isomorphism of operads.

### 3.3.3 Admissible graphs and corresponding differential forms

**Definition 13** *An admissible graph with parameters  $(n, m, k)$  (for  $n \geq 1$  and  $m \geq 0$ ) is a finite graph  $\Gamma$  such that*

- 1) *it has no multiple edges,*
- 2) *it contains no simple loops (edges connecting a vertex with itself),*
- 3) *it contains  $n + m$  vertices, numbered from 1 to  $n + m$ ,*
- 4) *it contains  $k$  edges, numbered from 1 to  $k$ ,*
- 5) *any vertex can be connected by a path with a vertex whose index is in  $\{1, \dots, n\}$ ,*
- 6) *any vertex with index in  $\{n + 1, \dots, n + m\}$  has valency (i.e. degree)  $\geq 3$ . greater than or equal to 3.*
- 7) *for every edge  $E$  of  $\Gamma$  we choose an orientation of this edge, i.e. we order the 2-element set of vertices to which  $E$  is attached.*
- 8) *if  $n = 1$  then the graph consists of just one vertex and has no edges, its parameters are  $(1, 0, 0)$ .*

Notice that although we endowed edges with orientations in 7), we treat in 1)-6) the graph  $\Gamma$  as an *unoriented* graph. The structure of an admissible graph is completely determined by the attachment map

$$\{1, \dots, k\} \longrightarrow \{(i, j) \mid 1 \leq i, j \leq n + m, i \neq j\}$$

from the set of edges to the set of ordered pairs of distinct vertices.

**Definition 14** Let  $\Gamma$  be an admissible graph. We define  $\omega_\Gamma$  to be the differential form on  $FM_d(n)$  given by the formula

$$\omega_\Gamma := (\pi_1)_* \circ \pi_2^* \left( \bigwedge_{\text{edges of } \Gamma} \text{Vol}_{S^{d-1}} \right) \quad (8)$$

where

$$\pi_1 : FM_d(n+m) \longrightarrow FM_d(n)$$

is the natural map, defined by forgetting the last  $m$  points in the configuration of  $(n+m)$  points in  $\mathbb{R}^d$ ,

$$\pi_2 : FM_d(n+m) \longrightarrow (FM_d(2))^k$$

is the product of forgetting maps  $FM_d(n+m) \longrightarrow FM_d(2) = S^{d-1}$  associated with the edges of  $\Gamma$  (i.e. with ordered pairs of indices in  $\{1, \dots, n+m\}$ ),

$$\text{Vol}_{S^{d-1}} \in \Omega^{d-1}(S^{d-1})$$

denotes the volume form on  $S^{d-1}$  invariant under the action of the rotation group  $SO(d, \mathbb{R})$  and normalized such that the total volume  $\int_{S^{d-1}} \text{Vol}_{S^{d-1}}$  is 1.

The degree of the form  $\omega_\Gamma$  is equal to

$$(d-1)k - dm = \dim(FM_d(2)^k) - (\dim(FM_d(n+m)) - \dim(FM_d(n))).$$

If one changes orientations of edges, or the enumeration of edges, or the enumeration of vertices with indices from  $n+1$  to  $n+m$ , one obtains the same form up to a sign. We leave as an easy exercise to the reader to write an explicit formula for this sign.

**Definition 15** For every  $n \geq 1$  we define  $\text{Graphs}_d(n)$  to be the  $\mathbb{Z}$ -graded vector space of all  $\mathbb{Q}$ -valued functions on the set of equivalence classes of admissible graphs with parameters  $(n, m, k)$ ,  $m$  and  $k$  arbitrary, such that if we change the enumeration of edges, or the enumeration of vertices with indices from  $n+1$  to  $n+m$ , then the value of the function will be multiplied by an appropriate sign as explained above. We define  $\mathbb{Z}$ -grading of a function concentrated on one given equivalence class  $[\Gamma]$  with parameters  $(n, m, k)$  as  $(dm - (d-1)k)$ .

Unfortunately, the forms  $\omega_\Gamma$  are not  $C^\infty$ -forms on the boundary of  $FM_d(n)$ . Still, for any  $\Gamma$  the integral of  $\omega_\Gamma$  over any semi-algebraic chain is absolutely convergent because the calculation of this integral reduces to the calculation the volume form over a compact semi-algebraic chain of the top degree in the product of spheres. The total volume is finite because the multiplicity of a semi-algebraic map is bounded. Thus the integral of the volume form is convergent.

From this follows that the form  $\omega_\Gamma$  gives a well-defined functional on  $\text{SemiAlgChains}(FM_d)$ . Turning things around, we can say that any semi-algebraic chain gives a functional on the set of equivalence classes of admissible graphs, i.e. an element of the real-valued version of  $\text{Graphs}_d(n)$ , the completed tensor product  $\mathbb{R} \hat{\otimes} \text{Graphs}_d(n)$ . The difference between the completed and the usual tensor product by  $\mathbb{R}$  will eventually disappear because we shall meet the operad  $H_*(FM_d)$  whose components are *finite-dimensional*.

**Lemma 3** *For any  $\Gamma$  the form  $d\omega_\Gamma$  (considered as a functional on semi-algebraic chains) is equal to the sum with appropriate signs of forms  $\omega_{\Gamma'}$  where the admissible graph  $\Gamma'$  is obtained from  $\Gamma$  by contraction of one edge.*

This lemma was proven in [K1] (lemma in 6.6) for the case  $d = 2$  and in [K2] (lemma 2.1) for the case  $d \geq 3$ .

Thus, the graded space  $\text{Graphs}_d(n)$  of functions on graphs carries a naturally defined differential, and forms a complex of vector spaces over  $\mathbb{Q}$ . Moreover, the restrictions of the forms  $\omega_\Gamma$  to irreducible components of the boundary of the manifolds  $FM_d(n)$  are finite linear combinations of products of analogous forms for simpler graphs. This means that  $\text{Graphs}_d$  is an operad of complexes, and the integration defines a homomorphism of operads of complexes of real vector spaces

$$\mathbb{R} \otimes \text{SemiAlgChains}(FM_d) \longrightarrow \mathbb{R} \hat{\otimes} \text{Graphs}_d$$

It is not obvious that this arrow is a quasi-isomorphism. This follows from an explicit calculation of the cohomology operad of the graph operad which we perform in the next subsection.

### 3.3.4 Forests and Tree complexes

**Definition 16** *An admissible graph is a forest iff it contains no nontrivial closed paths.*

It is easy to see that for non-forest graph  $\Gamma$  the differential  $d\omega_\Gamma$  is a linear combination of forms  $\omega_{\Gamma'}$  for non-forest graphs  $\Gamma'$ . Also, the restriction of  $\omega_\Gamma$  to irreducible components of the boundary of  $FM_d(n)$  is a linear combination of products  $\omega_{\Gamma_1} \times \omega_{\Gamma_2}$  where at least one of smaller graphs  $\Gamma_1, \Gamma_2$  is not a forest. All this implies that the following definition gives an operad:

**Definition 17** *The Operad  $\text{Forests}_d$  is a suboperad of  $\text{Graphs}_d$  consisting of functions vanishing on all non-forest graphs.*

A simple spectral sequence shows that the embedding

$$\text{Forests}_d \longrightarrow \text{Graphs}_d$$

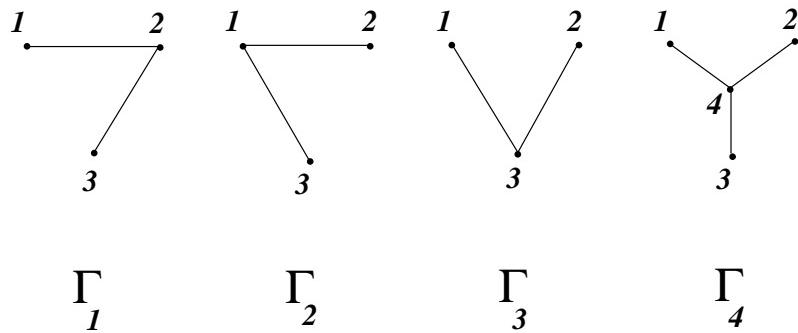
is a quasi-isomorphism. We shall not write it here explicitly, only mention the main idea of the calculation. Any graph which is not a forest contains a nonempty maximal subgraph

$\Gamma_{\text{core}}$  such that the valency (in  $\Gamma_{\text{core}}$ ) of each vertex of  $\Gamma_{\text{core}}$  is at least 2. The graph  $\Gamma$  is obtained from  $\Gamma_{\text{core}}$  by attaching several trees to vertices of  $\Gamma_{\text{core}}$ , and also a forest not connected with  $\Gamma_{\text{core}}$ . The desired result follows from the vanishing of the first term of the spectral sequence associated with the filtration by the number of vertices in  $\Gamma_{\text{core}}$  on the quotient complex  $\text{Graphs}_d(n)/\text{Forests}_d(n)$ .

Any admissible graph  $\Gamma$  with parameters  $(n, m, k)$  defines a partition of  $\{1, \dots, n\}$  into pieces corresponding to connected components of  $\Gamma$ . All the graphs  $\Gamma'$  which appear in the decomposition of  $d\omega_\Gamma$  (as in the lemma in 3.3.3) give the same partition of  $\{1, \dots, n\}$  as  $\Gamma$ . This implies that the forest complex  $\text{Forests}_d(n)$  splits naturally for any  $n$  into a direct sum of subcomplexes corresponding to partitions (i.e. equivalence relations) of the set  $\{1, \dots, n\}$ .

The subcomplex of  $\text{Forests}_d(n)$  associated with a partition is the tensor product of tree complexes over pieces of this partition, where *trees* are non-empty connected forests, as usual.

We show an example of the tree complex in the case of 3 vertices. There are 4 admissible trees with the parameter  $n$  equal to 3 (up to changing the enumeration of edges and orientations of edges):



The formula for the de Rham differential of forms  $\omega_{\Gamma_i}$  is

$$d\omega_{\Gamma_1} = d\omega_{\Gamma_2} = d\omega_{\Gamma_3} = 0, \quad d\omega_{\Gamma_4} = \pm\omega_{\Gamma_1} \pm \omega_{\Gamma_2} \pm \omega_{\Gamma_3} .$$

The tree complex is the *dual* complex to the complex spanned by the differential forms  $\omega_\Gamma$  for trees  $\Gamma$ . It is easy to see that the tree complex has the non-zero cohomology space only in the lowest degree. This implies that there is a natural map from the cohomology of the tree complex to the tree complex. Thus, we get a natural morphism from the cohomology of the forest complex to the forest complex. One can check that this map is a morphism of operads, and we get a quasi-isomorphism

$$H^*(\text{Forests}_d) \longrightarrow \text{Forests}_d$$

Comparing the cohomology of the forest complex with known results on the cohomology of configuration spaces, we see that

$$H^*(\text{Forests}_d) = H_*(C_d) = H_*(FM_d) .$$

Because we proved already that  $H^*(\text{Graphs}_d) = H^*(\text{Forests}_d)$ , we constructed a chain of quasi-isomorphisms as promised.

### 3.4 Application to deformation quantization

The theorems in 3.2 show that any Gerstenhaber algebra, i.e. an algebra over  $H_*(C_d)$ , can be canonically endowed with a  $d$ -algebra structure. We prove in this section the result of Tamarkin:

**Theorem 4** *Let  $A := \mathbb{R}[x_1, \dots, x_n]$  be the algebra of polynomials considered just as an associative algebra. Then the Hochschild complex  $\text{Hoch}(A)$  is quasi-isomorphic as 2-algebra to its cohomology*

$$B := H^*(\text{Hoch}(A)) = \text{space of polynomial polyvector fields on } \mathbb{R}^n$$

*considered as a Gerstenhaber algebra, hence a 2-algebra.*

As was mentioned in the introduction, this theorem implies the formality theorem from [K2].

It is well-known (Hochschild-Kostant-Rosenberg theorem) that the space of polyvector fields is equal to the cohomology of  $\text{Hoch}(A)$ , and the cup-product and the Lie bracket on  $B = H^*(\text{Hoch}(A))$  are the usual cup-product and the Schouten-Nijenhuis bracket on polyvector fields, respectively. From the general formalism of deformation theory it follows that if  $\text{Hoch}(A)$  is *not* quasi-isomorphic to  $B$  then there will be a *non-zero* obstruction element  $\gamma_{\text{obstr}}$  in the first cohomology group of the deformation complex  $\text{Def}(B)$  of 2-algebra  $B$ . Moreover, there exists an  $\text{Aff}(\mathbb{R}^n)$ -invariant splitting of  $\text{Hoch}(A)$  into the direct sum of  $B$  and a splitted contractible complex (see arguments in [K1], 4.6.1.1). By the lemma in 2.6, this splitting induces a structure of 2-algebra on  $B$ , and this structure is  $\text{Aff}(\mathbb{R}^n)$ -invariant. It implies that  $\gamma_{\text{obstr}} \in H^1(\text{Def}(B))$  is also  $\text{Aff}(\mathbb{R}^n)$ -invariant.

Let us calculate the cohomology of the deformation complex of the 2-algebra  $B$ . As a first approximation, we calculate the Hochschild cohomology of  $B$ :

**Theorem 5** *The Hochschild complex of  $B$  is quasi-isomorphic to  $\mathbb{R}^1$  placed in degree 0.*

In order to prove it we need an additional result concerning the Hochschild cohomology of  $d$ -algebras with trivial Lie bracket. By results from 3.2 we can speak about algebras over  $H_*(C_d)$  instead of  $d$ -algebras.

**Lemma 4** *Let  $d \geq 2$  be an integer. Consider the algebra of polynomials in a finite number of  $\mathbb{Z}$ -graded variables*

$$\mathbb{R}[x_1, \dots, x_N], \quad \deg(x_i) = d_i \in \mathbb{Z}$$

*as an algebra over the operad  $H_*(C_d)$ , endowing it with zero differential and with the vanishing Lie bracket. Then the Hochschild cohomology of this algebra is, as  $\mathbb{Z}$ -graded vector space, the same as the algebra of polynomials in the doubled set of variables*

$$\mathbb{R}[x_1, \dots, x_N, y_1, \dots, y_N], \quad \deg(x_i) = d_i, \quad \deg(y_i) = d - d_i$$

The statement of this lemma is similar to the classical Hochschild-Kostant-Rosenberg calculation of the Hochschild cohomology of the algebra of polynomials considered as an associative algebra (i.e. as 1-algebra). We shall not give here the proof of this lemma.

In general, if  $\mathcal{O}(X)$  is the algebra of functions on a smooth  $\mathbb{Z}$ -graded algebraic supermanifold  $X$ , then the Hochschild cohomology of  $\mathcal{O}(X)$  considered as a  $d$ -algebra, coincides with the algebra of functions on the total space of the twisted by  $[d]$  cotangent bundle to  $X$ :

$$H^*(\text{Hoch}(\mathcal{O}(X))) = \mathcal{O}(T^*[d]X) .$$

Applying the above lemma to the following  $H_*(C_2)$ -algebra with *vanishing* Lie bracket

$$B_0 := \mathbb{R}[x_1, \dots, x_n, \xi_1, \dots, \xi_n], \deg(x_i) = 0, \deg(\xi_i) = +1$$

we get the Hochschild cohomology

$$C_0 := \mathbb{R}[(x_i), (\xi_i), (\eta_i), (y_i)]_{1 \leq i \leq n},$$

$$\deg(x_i) = 0, \deg(\xi_i) = \deg(\eta_i) = +1, \deg(y_i) = +2 .$$

Our 2-algebra  $B$  is obtained from  $B_0$  by switching on the Lie bracket:

$$[x_i, x_j] = [\xi_i, \xi_j] = 0, [x_i, \xi_j] = \delta_{ij} .$$

It is easy to see that this produces the following differential on  $C_0$ :

$$d_1 := \sum_{i=1}^n (\eta_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial \xi_i})$$

The cohomology of the complex  $(C_0, d_1)$  is equal to the de Rham cohomology of  $\mathbb{R}^n$ , i.e. to  $\mathbb{R}$  placed in degree 0. What we calculated is only the first term in a spectral sequence, but it is clear that higher differentials are zero because there is no space for them. This proves our theorem on the Hochschild cohomology of  $B = \text{Hoch}[x_1, \dots, x_n]$ :

$$H^*(\text{Hoch}(\text{Hoch}(\mathbb{R}[x_1, \dots, x_n]))) = \mathbb{R} . \quad (9)$$

Now we are ready to calculate the first cohomology of the deformation complex of  $B$ . Remind (see 2.7) that there is a short exact sequence

$$0 \longrightarrow B[1] \longrightarrow \text{Def}(B) \longrightarrow \text{Hoch}(B)[2] \longrightarrow 0 .$$

Passing from this exact sequence to the level of the cohomology we get a long exact sequence

$$\dots \longrightarrow H^{i+1}(\text{Hoch}(B)) \longrightarrow B^{i+1} \longrightarrow H^i(\text{Def}(B)) \longrightarrow H^{i+2}(\text{Hoch}(B)) \longrightarrow \dots$$

which for  $i = 1$  says that

$$H^1(\text{Def}(B)) = B^2 = \Gamma(\mathbb{R}^d, \wedge^2 T) = \{ \text{polynomial bivector fields on } \mathbb{R}^d \}$$

Then the argument goes as was sketched in the introduction. The explicit quasi-isomorphism between the space of polyvector fields and the Hochschild complex of the associative algebra  $A = \mathbb{R}[x_1, \dots, x_n]$  can be made invariant under the action of the group of affine transformations. There is no non-zero  $\text{Aff}(\mathbb{R}^n)$ -invariant bi-vector fields on  $\mathbb{R}^n$ . Thus the first non-trivial obstruction  $\gamma_{\text{obstr}}$  to the existence of a quasi-isomorphism between  $B$  and  $\text{Hoch}(A)$  cannot exist, and  $B$  is quasi-isomorphic to  $\text{Hoch}(A)$ .

## 4 Grothendieck-Teichmüller group action on quantizations

### 4.1 Integrals in deformation quantization

There are several situations in deformation quantization where coefficients in formulas are given by explicit integrals.

A) In Drinfeld's study of classical Knizhnik-Zamoldchikov equations (see [D]), a formal series in two non-commuting variables appears, called an *associator*. The coefficients in the explicit formula for the associator are iterated integrals

$$I_{\epsilon_1, \dots, \epsilon_n} := \int_{0 < t_1 < \dots < t_n < 1} \omega_{\epsilon_1}(t_1) \wedge \dots \wedge \omega_{\epsilon_n}(t_n) \quad (10)$$

where  $\epsilon_i \in \{0, 1\}$ ,  $\epsilon_0 = 1, \epsilon_n = 0$ ,  $\omega_0(t) = dt/t$ ,  $\omega_1(t) = dt/(1-t)$ .

Among these numbers there are values of the Riemann zeta-function at positive integers,  $\zeta(n) = I_{1,0,0,\dots,0}$  for a  $n$ -dimensional integral. In general, the integrals  $I_{\epsilon_1, \dots, \epsilon_n}$  can be identified with so called *multiple zeta-values* (see [Z]).

B) The same class of numbers appears in the Etingof-Kazhdan quantization of Poisson-Lie algebras because it was based on Drinfeld's work (see [EK]).

C) Tamarkin uses the Drinfeld associator and Etingof-Kazhdan results from [EK] in his proof of the formality of  $\text{Chains}(C_2) \otimes \mathbb{C}$  (see [T2]).

D) Tamarkin uses also the Drinfeld associator in the new proof of the formality theorem in deformation quantization (see [T1]).

E) In my construction of quantization of Poisson manifolds (see [K1]) other types of integrals were used. These integrals are real-valued and are expressed by:

$$\int_{U_{n,m}} \bigwedge_{k=1}^{m+2n-2} \alpha(z_{i_k}, z_{j_k}), \quad m \geq 0, n \geq 1 \quad (11)$$

where the domain of integration  $U_{n,m}$  is

$$\begin{aligned} &\{(z_1, \dots, z_{n+m}) \in \mathbb{C}^{n+m} \mid z_1, \dots, z_m \in \mathbb{R}, z_1 < \dots < z_m; \\ &\quad \Im(z_{m+1}), \dots, \Im(z_{m+n}) > 0; z_i \neq z_j; z_{m+n} = \sqrt{-1}\} \end{aligned}$$

The form  $\alpha(z, w)$  is

$$\frac{1}{4\pi i} d \log \left( \frac{(z-w)(z-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})} \right). \quad (12)$$

The range of indices  $i_k \neq j_k$  is  $\{1, \dots, n+m\}$  for all  $k \in \{1, \dots, m+2n-2\}$ .

F) Recently I realized that one can modify the above formulas, replacing  $\alpha$  by

$$\alpha_{new}(z, w) := d \log \left( \frac{(z-w)}{(\bar{z}-w)} \right) \quad (13)$$

and dividing the integral by  $(2\pi i)^n$ . In this way one gets complex-valued integrals, and all identities proven in [K2] remain true.

G) The last example is the use of integrals in the proof of formality of  $\text{Chains}(C_2) \otimes \mathbb{R}$  presented in section 3.3.

I claim that all these integrals are closely related, and their use is probably unavoidable.

## 4.2 Torsors

In all the above situations A)-G), we are constructing *identifications* (up to a homotopy) between certain pairs of algebraic structures. For example, in deformation quantization we construct an isomorphism in the homotopy category of Lie algebras between the shifted by [1] Hochschild complex of  $\mathbb{R}[x_1, \dots, x_n]$ , and the graded Lie algebra of polyvector fields on  $\mathbb{R}^n$ .

Let  $C$  be any category and  $\mathcal{E}, \mathcal{F}$  be two isomorphic objects in this category. The set of isomorphisms

$$\text{Iso}(\mathcal{E}, \mathcal{F})$$

is a non-empty set on which the group  $\text{Aut}(\mathcal{E})$  acts simply transitively. The group  $\text{Aut}(\mathcal{F})$  acts also simply transitively on it, and the two actions commute. One can encode all these structures in a single map

$$\text{Iso}(\mathcal{E}, \mathcal{F})^3 \longrightarrow \text{Iso}(\mathcal{E}, \mathcal{F}), \quad (a, b, c) \mapsto a \circ b^{-1} \circ c \quad (14)$$

**Definition 18** A torsor is a non-empty set  $X$  endowed with a map  $X \times X \times X \longrightarrow X$  satisfying the same identities as maps  $(a, b, c) \mapsto ab^{-1}c$  in groups.

A torsor is the same as a principal homogeneous space over a group. One can give a more transparent definition of essentially the same structure:

**Definition 19** A torsor is a category  $C$  with only two objects  $\text{Ob}_1$  and  $\text{Ob}_2$ , such that all the morphisms in  $C$  are invertible and the objects  $\text{Ob}_1$  and  $\text{Ob}_2$  are equivalent.

If  $X$  is a torsor (by Definition (18)) then one has two groups acting simply transitively on  $X$ . Any element  $x \in X$  gives an identification between these two groups.

### 4.2.1 Pro-algebraic torsors

In each of the cases listed in 4.1 in order to construct an isomorphism one is brought to solve an infinite system of quadratic equations with integer coefficients. For example, in deformation quantization we need to choose weights for all finite graphs in order to get a quasi-isomorphism between two homotopy Lie algebras. This implies that the torsor of isomorphisms should be considered as an infinite-dimensional algebraic variety over  $\mathbb{Q}$ , not only as a set. We denote the torsor of isomorphisms considered as an algebraic variety by

$$\underline{\mathsf{Iso}}(\mathcal{E}, \mathcal{F})$$

Each of the two groups acting on the torsor is a projective limit of finite-dimensional affine algebraic groups over  $\mathbb{Q}$ . The algebra of functions  $\mathcal{O}(T)$  on a pro-algebraic torsor  $T$  is a generalization of a Hopf algebra. Namely, it is a commutative associative unital algebra over  $\mathbb{Q}$  together with the structure map

$$\mathcal{O}(T) \mapsto \mathcal{O}(T) \otimes \mathcal{O}(T) \otimes \mathcal{O}(T) \quad (15)$$

which is a homomorphism of algebras, and satisfies relations dual to the defining relations for set-theoretic torsors. We call this map the *triple coproduct* in  $\mathcal{O}(T)$ .

The integrals in situations A)-G) from 4.1 give solutions to systems of quadratic equations in complex numbers, i.e. they give a homomorphism of algebras over  $\mathbb{Q}$

$$\mathcal{O}(T) \longrightarrow \mathbb{C}$$

where the pro-algebraic torsor  $T$  depends on the concrete problem which we consider. In the next subsection we are going to describe a countable-dimensional commutative algebra  $P$  over  $\mathbb{Q}$  such that the homomorphism from  $\mathcal{O}(T)$  to  $\mathbb{C}$  is the composition of homomorphisms  $\mathcal{O}(T) \longrightarrow P$  and  $P \longrightarrow \mathbb{C}$ .

### 4.3 Periods, motivic Galois group, motives

Periods are integrals of algebraic differential forms with algebraic coefficients. The following numbers are periods:

$$1, \sqrt{2}, i = \sqrt{-1}, \pi, \log(2), \\ \zeta(3) = \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3}, \text{ elliptic integral } \int_1^2 \sqrt{x^3 + 1} dx, \dots$$

We shall need a more precise definition of periods. Let  $X$  be a smooth algebraic variety of dimension  $d$  defined over  $\mathbb{Q}$ ,  $D \subset X$  be a divisor with normal crossings (i.e. locally  $D$  looks like a collection of coordinate hypersurfaces),  $\omega \in \Omega^d(X)$  be an algebraic differential form on  $X$  of top degree ( $\omega$  is automatically closed), and  $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$  be a (homology class of a) singular chain on the complex manifold  $X(\mathbb{C})$  with boundary on the divisor  $D(\mathbb{C})$ . With these data one associates the integral  $\int_\gamma \omega \in \mathbb{C}$ . We say that this number is the period of the quadruple  $(X, D, \omega, \gamma)$ . One can always reduce convergent

integrals of algebraic forms over semi-algebraic sets defined over the field algebraic numbers  $\overline{\mathbb{Q}}$  to the form as above, using the functor of the restriction of scalars to  $\mathbb{Q}$  and the resolution of singularities in characteristic zero.

Usual tools for proving identities between integrals are the change of variables and the Stokes formula. Let us formalize them for the case of periods.

**Definition 20** *The space  $P_+$  of effective periods is defined as a vector space over  $\mathbb{Q}$  generated by the symbols  $[(X, D, \omega, \gamma)]$  representing equivalence classes of quadruples as above, modulo the following relations:*

- 1) (linearity)  $[(X, D, \omega, \gamma)]$  is linear in both  $\omega$  and  $\gamma$
- 2) (change of variables) If  $f : (X_1, D_1) \rightarrow (X_2, D_2)$  is a morphism of pairs defined over  $\mathbb{Q}$ ,  $\gamma_1 \in H_d(X_1(\mathbb{C}), D_1(\mathbb{C}); \mathbb{Q})$  and  $\omega_2 \in \Omega^d(X_2)$  then

$$[(X_1, D_1, f^*\omega_2, \gamma_1)] = [(X_2, D_2, \omega_2, f_*(\gamma_1))]$$

3) (Stokes formula) Denote by  $\tilde{D}$  the normalization of  $D$  (i.e. locally it is the disjoint union of irreducible components of  $D$ ), the variety  $\tilde{D}$  containing a divisor with normal crossing  $\tilde{D}_1$  coming from double points in  $D$ . If  $\beta \in \Omega^{d-1}(X)$  and  $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$  then

$$[(X, D, d\beta, \gamma)] = [(\tilde{D}, \tilde{D}_1, \beta|_{\tilde{D}}, \partial\gamma)]$$

where  $\partial : H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}) \rightarrow H_{d-1}(\tilde{D}(\mathbb{C}), \tilde{D}_1(\mathbb{C}); \mathbb{Q})$  is the boundary operator.

It is conjectured in number theory that the evaluation homomorphism  $P_+ \rightarrow \mathbb{C}$  is a monomorphism, i.e. all identities between periods can be proved using standard rules only. For example, the fact that number  $\pi$  is transcendental follows from this conjecture.

The effective periods form an algebra because the product of integrals is again an integral (Fubini formula). The field of algebraic numbers  $\overline{\mathbb{Q}} \subset \mathbb{C}$  can be considered as a subalgebra over  $\mathbb{Q}$  of the algebra  $P_+$ . An algebraic number  $x \in \overline{\mathbb{Q}}$  which solves a polynomial equation  $P = 0$ ,  $P \in \mathbb{Q}[t]$ , is the period of 0-dimensional variety  $X \subset \mathbb{A}_{\mathbb{Q}}^1$  defined by the equation  $P = 0$ . The number  $x$  gives a complex point of  $X$ , i.e. a 0-chain. The standard coordinate  $t$  on the affine line  $\mathbb{A}_{\mathbb{Q}}^1$  gives a zero-form after restriction to  $X$  whose pairing with  $x \in X(\mathbb{C}) \subset \mathbb{A}_{\mathbb{Q}}^1(\mathbb{C}) = \mathbb{C}$  is tautologically equal to  $x$ .

It is convenient to extend the algebra of effective periods to a larger algebra  $P$  by inverting formally the element whose evaluation in  $\mathbb{C}$  is  $2\pi i$ . Informally, we can write that the whole algebra of periods  $P$  is  $P_+[(2\pi i)^{-1}]$ .

The algebra  $P$  is an infinitely generated algebra over  $\mathbb{Q}$ , but as any algebra it is an inductive limit of finitely-generated subalgebras. This means that  $\text{Spec}(P)$  is a projective limit of finite-dimensional affine schemes over  $\mathbb{Q}$ . We claim that  $\text{Spec}(P)$  carries a natural structure of a pro-algebraic torsor over  $\mathbb{Q}$ .

The formula for the structure map  $\Delta : P \rightarrow P \otimes P \otimes P$  can easily be written in terms of period matrices of individual algebraic varieties. Namely, let  $(P_{ij})$  be the period matrix of an algebraic variety consisting of pairings between classes running through a basis in  $H_*(X(\mathbb{C}), \mathbb{Q})$  and a basis in  $H_{\text{de Rham}}^*(X)$ . More generally, one should consider homology

and cohomology of pairs of algebraic varieties over  $\mathbb{Q}$ . It follows from several results in algebraic geometry that the period matrix is a square matrix with entries in  $P_+$ , and determinant in  $\sqrt{\mathbb{Q}^\times} \cdot (2\pi i)^{\mathbb{Z}_{\geq 0}}$ . This implies that the inverse matrix has coefficients in the extended algebra  $P = P_+[(2\pi i)^{-1}]$ .

**Definition 21** *The triple coproduct in  $P$  is defined by*

$$\Delta(P_{ij}) := \sum_{k,l} P_{ik} \otimes (P^{-1})_{kl} \otimes P_{lj} \quad (16)$$

for any period matrix  $(P_{ij})$ .

We show how to calculate the triple coproduct in a simple example. Let  $X := \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}$  be the affine line with deleted point 0, and  $D := \{1, 2\} \subset X$  be a divisor in  $X$ . The first homology group of pair (relative homology)

$$H_1(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}) = H_1(\mathbb{C} \setminus \{0\}, \{1, 2\}; \mathbb{Q})$$

is two-dimensional and is generated by two chains: a small closed anti-clockwise oriented path around 0, and the interval  $[1, 2]$ . The algebraic de Rham cohomology group  $H_{\text{de Rham}}^1(X, D)$  is also two-dimensional, and is generated by the 1-forms  $dt$  and  $dt/t$  where  $t$  is the standard coordinate on  $X \subset \mathbb{A}_{\mathbb{Q}}^1$ . The period matrix is

$$\begin{pmatrix} 1 & \log(2) \\ 0 & 2\pi i \end{pmatrix}. \quad (17)$$

From this one can deduce that

$$\Delta(2\pi i) = 2\pi i \otimes \frac{1}{2\pi i} \otimes 2\pi i,$$

$$\Delta(\log(2)) = (\log(2) \otimes \frac{1}{2\pi i} \otimes 2\pi i) - (1 \otimes \frac{\log(2)}{2\pi i} \otimes 2\pi i) + (1 \otimes 1 \otimes \log(2)).$$

It is not clear why the definition given above is consistent, because it is not obvious why the triple coproduct preserves the defining relations in  $P$ . This follows more or less automatically from the following result:

**Theorem 6 (M. Nori)** *Algebra  $P$  over  $\mathbb{Q}$  is the algebra of functions on the pro-algebraic torsor of isomorphisms between two cohomology theories, the usual topological cohomology theory*

$$H_{\text{Betti}}^* : X \mapsto H^*(X(\mathbb{C}), \mathbb{Q})$$

and the algebraic de Rham cohomology theory

$$H_{\text{de Rham}}^* : X \mapsto \mathbf{H}^*(X, \Omega_X^*)$$

The motivic Galois group in Betti realization  $G_{M,\text{Betti}}$  is defined as the pro-algebraic group acting on  $\text{Spec}(P)$  from the side of Betti cohomology. Analogously one defines the de Rham version  $G_{M,\text{deRham}}$ . The category of motives is defined as the category of representations of the motivic Galois group. It does not matter which realization we choose because the categories for both realizations can be canonically identified with each other. Here is the “symmetric” definition of the category of motives:

**Definition 22** *The symmetric monoidal category of motives over  $\mathbb{Q}$  with coefficients in  $\mathbb{Q}$  is defined as the category of vector bundles on  $\text{Spec}(P)$  endowed with two commuting actions of the motivic Galois groups  $G_{M,\text{Betti}}$  and  $G_{M,\text{deRham}}$ .*

The following elementary definition gives a category equivalent to the category of motives:

**Definition 23** *A framed motive of rank  $r \geq 0$  is an invertible  $(r \times r)$ -matrix  $(P_{ij})_{1 \leq i,j \leq r}$  with coefficients in the algebra  $P$ , satisfying the equation*

$$\Delta(P_{ij}) = \sum_{k,l} P_{ik} \otimes (P^{-1})_{kl} \otimes P_{lj} \quad (18)$$

for any  $i, j$ . The space of morphisms from one framed motive to another, corresponding to matrices

$$P^{(1)} \in GL(r_1, P), \quad P^{(2)} \in GL(r_2, P),$$

is defined as

$$\{T \in \text{Mat}(r_2 \times r_1, \mathbb{Q}) \mid TP^{(1)} = P^{(2)}T\}$$

The cohomology groups of varieties over  $\mathbb{Q}$  can be considered as objects of the category of motives. From comparison isomorphisms in algebraic geometry follows that there are also  $l$ -adic realizations motives, on which the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \text{Aut}(\overline{\mathbb{Q}})$  acts.

The usual Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is a profinite group, projective limit of finite groups. It is the image of the motivic Galois group  $G_{M,\text{Betti}}$  in motives coming from 0-th cohomology groups of schemes defined over  $\mathbb{Q}$ . It can be described also as follows: the subalgebra  $\overline{\mathbb{Q}}$  of  $P$  is closed under the triple coproduct,

$$\Delta_{|\overline{\mathbb{Q}}} : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

and its spectrum is a quotient torsor of  $P$  on which  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts simply transitively.

The definition of the category of motives given above is a natural generalization of the “folklore” definition of the category of pure motives given for example in [S]. There is an elaborated hypothetical picture of motives in number theory, from which follows that the category of motives as defined here is exactly what is expected. The definition given here is equivalent to the one advocated by M. Nori in [N]. Specialists in motives consider this definition as “cheap”, and expect in the future something more elaborate and not directly referring to explicit realization functors (like Betti realization etc.).

The advantage of the definition given here is that it does not assume the validity of any conjecture, and is directly applicable to the present study of operads in deformation quantization.

#### 4.4 Grothendieck-Teichmüller group

All the integrals appearing in situations listed in 4.1 share several common features. The corresponding motives are so called *mixed Tate motives*, which means in particular that the period matrix is upper-triangular in certain bases, with integral powers of  $(2\pi i)$  on the diagonal (like in the example in the previous subsection). Also these motives are *unramified over  $\text{Spec}(\mathbb{Z})$* . This property can be expressed in terms of  $l$ -adic representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and can be verified in concrete situations by checking that certain discriminants are equal to 0, 1 or  $-1$ . The period matrix (17) above, is *not* unramified. Namely, it is ramified at prime 2.

We denote by  $P_{\mathbb{Z}, \text{Tate}}$  the subalgebra of  $P$  generated by  $(2\pi i)^{\pm 1}$  and by periods of mixed Tate motives unramified over  $\text{Spec}(\mathbb{Z})$ . There is a conjectural picture for it:

**Conjecture 2** *The quotient of the motivic Galois group (in the de Rham realization) acting simply transitively on  $\text{Spec}(P_{\mathbb{Z}, \text{Tate}})$  is a pro-solvable connected group over  $\mathbb{Q}$ , an extension of the multiplicative groups scheme  $\mathbf{G}_m = GL(1)$  by a pro-nilpotent group whose Lie algebra is free and generated by elements of weights  $3, 5, 7, \dots$  (one element in each odd weight  $\geq 3$ ) with respect to the adjoint action of  $\mathbf{G}_m$ .*

This conjecture follows from general Beilinson conjectures on motives and  $K$ -theory (see [Ne]).

There are two other conjectures concerning  $P_{\mathbb{Z}, \text{Tate}}$ .

**Conjecture 3**  *$P_{\mathbb{Z}, \text{Tate}}$  is the subalgebra of  $P$  generated by  $(2\pi i)^{\pm 1}$  and by periods whose evaluation in  $\mathbb{C}$  are integrals  $I_{\epsilon_1, \dots, \epsilon_n}$  which appear in Drinfeld's associator (see 4.1.A).*

There are two reasons for this conjecture. First of all, if we believe in the picture of  $P_{\mathbb{Z}, \text{Tate}}$  explained above, there are  $O(c^n)$  linearly independent effective unramified Tate motives of weight  $n$ , where  $c = 1.32471\dots$  is the positive root of the equation  $x^3 = x + 1$ . On the other hand, there are  $2^{n-2}$  integrals  $I_{\epsilon_1, \dots, \epsilon_n}$ , and with high probability (because  $c < 2$ ) there are enough integrals to span the whole algebra  $P_{\mathbb{Z}, \text{Tate}}$ . The second reason is also probabilistic, computer experiments confirm that the Poincaré series of the algebra generated by integrals graded by weights  $w(I_{\epsilon_1, \dots, \epsilon_n}) := n$  equals to the expected series  $1/(1 - t^2 - t^3)$  up to  $O(t^{13})$ .

The next conjecture concerns the so called Grothendieck-Teichmüller group  $GT$  (see [D]).

**Conjecture 4** *The quotient of the motivic Galois group acting simply transitively on the spectrum of the subalgebra of  $P$  generated by  $(2\pi i)^{\pm 1}$  and integrals  $I_{\epsilon_1, \dots, \epsilon_n}$ , coincides with the group  $GT$ .*

The group  $GT$  is a pro-algebraic group over  $\mathbb{Q}$ , an extension of  $\mathbf{G}_m$  by a pro-nilpotent group. One of the definitions of  $GT$  is as the group of automorphisms of the tower of pro-nilpotent completions of pure braid groups  $\pi_1(\mathbb{C}^n \setminus \text{Diag})$  for all  $n$ . The group of automorphisms of these pro-nilpotent completions coincides with the group of automorphisms of the tower of *rational homotopy types* of classifying spaces of these groups. Classifying spaces are  $(\mathbb{C}^n \setminus \text{Diag})_{n \geq 2}$ , configuration spaces of  $\mathbb{R}^2 = \mathbb{C}$ . By general reasons the motivic Galois group acts on the tower of rational homotopy types of  $(\mathbb{C}^n \setminus \text{Diag}) = (\mathbb{A}_{\mathbb{Q}}^n \setminus \text{Diag})(\mathbb{C})$ , thus it maps to  $GT$ . Moreover, it is easy to see that the periods which appear in the image of this action are exactly the numbers  $I_{\epsilon_1, \dots, \epsilon_n}$ .

A priori, there is no reason for the tower of Malcev completed braid groups to have a non-trivial automorphism. Number theory provides (via motivic Galois group) a supply of such automorphisms, and the conjecture 4 above says that there is no other automorphism.

The group  $GT$  maps to the group of automorphisms in homotopy sense of the operad  $\text{Chains}(C_2)$ . Moreover, it seems to coincide with  $\text{Aut}(\text{Chains}(C_2))$  when this operad is considered as an operad not of complexes but of differential graded cocommutative coassociative coalgebras (strictly speaking there is no coproduct on singular cubical chains, but one can overcome this technical problem).

From now on we shall assume the validity of conjectures 2,3, and 4 and identify the group acting simply transitively on  $P_{\mathbb{Z}, \text{Tate}}$  with the Grothendieck-Teichmüller group  $GT$ .

#### 4.5 Conjectures about torsors

Remind that the integrals described in situations A)-G) in 4.1 give complex points of the corresponding torsors, i.e. homomorphisms of algebras over  $\mathbb{Q}$

$$\mathcal{O}(T) \longrightarrow \mathbb{C} .$$

Moreover, the values of the integrals in all these cases are periods, and we proved that quadratic equations are satisfied using only the Stokes formula. This implies that we have in fact a homomorphism

$$\mathcal{O}(T) \longrightarrow P_{\mathbb{Z}, \text{Tate}} (\hookrightarrow P) \tag{19}$$

In terms of pro-algebraic affine schemes, this means that we have a map

$$\text{Spec}(P_{\mathbb{Z}, \text{Tate}}) = \underline{\text{Iso}}_{\mathbb{Z}, \text{Tate}}(H_{\text{de Rham}}^*, H_{\text{Betti}}^*) \longrightarrow T \tag{20}$$

between two torsors considered just as pro-algebraic schemes. For example, in the formality theorem (cases D),E),F)) the torsor  $T$  is the torsor of isomorphisms between two homotopy Lie algebras

$$T = \underline{\text{Iso}}(T_{\text{poly}}^*, D_{\text{poly}}^*)$$

(see [K1] for a precise definition of  $T_{\text{poly}}^*$  and  $D_{\text{poly}}^*$ ; one can safely replace  $D_{\text{poly}}^*$  by the shifted by [1] Hochschild complex of the algebra  $A := \mathbb{R}[x_1, \dots, x_n]$ , and  $T_{\text{poly}}^*$  by the algebra of polynomial polyvector fields on  $\mathbb{R}^d$ ). It is natural to ask whether the constructed map is in fact a map of torsors.

**Conjecture 5** *In cases A),B),C),D),F) the map from  $P$  to the corresponding torsor of isomorphisms is a map of torsors.*

I checked that the generator of  $\mathbb{Z}/2\mathbb{Z} \subset G_{M,\text{Betti}}$  (acting via complex conjugation on  $X(\mathbb{C})$  where  $X$  is defined over  $\mathbb{Q}$ ) corresponds to the natural involution of the Hochschild complex  $C^*(A, A)$  for  $A = \mathbb{R}[x_1, \dots, x_n]$ , the Hochschild complex being considered as a dg-Lie algebra. The involution acts on  $\phi \in C^n(A, A)$  as

$$\phi \mapsto \bar{\phi}, \quad \bar{\phi}(a_1 \otimes \dots \otimes a_n) := (-1)^{(n+1)(n+2)/2} \phi(a_n \otimes \dots \otimes a_1).$$

In terms of deformation theory this corresponds to the passage from a product  $\star$  to the *opposite* product

$$a \bar{\star} b := b \star a$$

Also, the group  $\mathbf{G}_m$  acts on  $\text{Spec}(P_{\mathbb{Z}, \text{Tate}})$  from the de Rham side, and also on  $\underline{\text{Iso}}(T_{\text{poly}}^*, D_{\text{poly}}^*)$  from the side of  $T_{\text{poly}}^*$  rescaling polyvector fields according to  $\mathbb{Z}$ -grading. Again, I checked that the two actions of  $\mathbf{G}_m$  are compatible.

In cases the E),G) not listed in the conjecture, the values of the integrals are *real* numbers. It seems that the map between torsors does not respect torsor structure. Moreover, I think that it is equal to the composition of a map of torsors with a certain universal map of schemes  $P_{\mathbb{Z}, \text{Tate}} \rightarrow P_{\mathbb{Z}, \text{Tate}}$  which is *not* a map of torsors.

## 4.6 Incarnations of the Grothendieck-Teichmüller group

I present here two examples in which one can see the action of  $GT$  on deformation quantizations. Proofs will appear elsewhere.

**Theorem 7** *Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over  $\mathbb{R}$ . In the Duflo-Kirillov isomorphism (see [K1]) between the center of the universal enveloping algebra  $U\mathfrak{g}$ , and the algebra of invariant polynomials on  $\mathfrak{g}^*$ , one can replace the formal series*

$$F(x) = \sqrt{\frac{e^{x/2} - e^{-x/2}}{x}}$$

by the product

$$F_{\text{new}}(x) = F(x) \cdot \exp \left( \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \right)$$

where  $a_1, a_3, a_5, \dots$  are arbitrary constants. For any such choice one gets again an isomorphism compatible with products.

In particular, one can replace  $F$  by the following entire function

$$F_{\text{nice}}(x) = \frac{1}{\Gamma(\frac{x}{2\pi i} + 1)}$$

for the choice

$$a_1 = \frac{\text{Euler constant}}{2\pi i} = \frac{0.57721...}{2\pi i}, \quad a_3 = \frac{\zeta(3)}{3(2\pi i)^3}, \quad a_5 = \frac{\zeta(5)}{5(2\pi i)^5}, \dots$$

The Euler constant is (probably) not a period, and enters the formula only for aesthetical reasons. The terms with coefficients  $a_3, a_5, \dots$  appear because of the action of  $GT$ .

As a corollary to Theorem 7 above we have the following

**Theorem 8** *If  $\mathfrak{g}$  is a finite-dimensional Lie algebra, then the differential operators on  $\mathfrak{g}^*$  with constant coefficients which are Fourier transforms of the following polynomials on  $\mathfrak{g}$ :*

$$P_{2k+1}(\gamma) := \text{Trace}(\text{ad}(\gamma)^{2k+1}), \quad k \geq 0$$

*act on the subalgebra of  $\text{ad}(\mathfrak{g})$ -invariant polynomials as derivations, i.e. satisfy the Leibniz rule.*

Unfortunately, as I learned from M. Duflo, for all finite-dimensional Lie algebras, operators such as in those considered in Theorem 7 above, when restricted to  $\text{Sym}(\mathfrak{g})^\mathfrak{g}$  are all equal to zero. Thus, we do not get anything visible here. Nevertheless, my result works also for Lie algebras in rigid symmetric monoidal categories, for example for a finite-dimensional Lie superalgebra. There is a chance that one can find non-trivial examples there.

Another incarnation of the motivic Galois group is in a sense Fourier dual to the previous one. Namely, let  $X$  be a complex manifold (or a smooth algebraic variety over a field of characteristic zero). Define the Hochschild cohomology of  $X$  as the following graded commutative associative algebra:

$$HH^k(X) := \bigoplus_{i+j=k} H^i(X, \wedge^j T_X) .$$

The product on  $HH^*(X)$  is given by the usual cup-product of polyvector fields and of cohomology classes. Every element of the *Hodge* cohomology

$$\bigoplus_{i,j} H^i(X, \wedge^j T_X^*)$$

gives a linear operator on  $HH^*(X)$ . It comes from the cup-product in cohomology and from the convolution operator

$$\wedge^a T_X^* \otimes \wedge^b T_X \longrightarrow \wedge^{b-a} T_X$$

acting fiberwise on the level of bundles. Operators corresponding to elements in  $H^i(X, \wedge^i T_X^*)$  act on  $HH^*(X)$  preserving the  $\mathbb{Z}$ -grading.

One can construct a version of characteristic classes of vector bundles on  $X$  with values in the diagonal part  $\bigoplus_{i \geq 0} H^i(X, \wedge^i T_X^*)$  of the Hodge cohomology. This can be done using the Atiyah class of a vector bundle  $\mathcal{E}$  taking values in  $H^1(X, T_X^* \otimes \text{End}(\mathcal{E}))$ . This class is the class of the extension of  $\mathcal{E}$  by  $T_X^* \otimes \mathcal{E}$  given by the bundle of 1-jets of sections of  $\mathcal{E}$ . The traces of powers of the Atiyah class give characteristic classes associated with monomial symmetric functions, and other characteristic classes are polynomials in basic ones.

**Theorem 9** *Operators on  $HH^*(X)$  corresponding to odd components*

$$ch_{2k+1}(T_X) \in H^{2k+1}(X, \wedge^{2k+1} T_X^*)$$

of the Chern character  $ch(T_X)$  of the tangent bundle, are derivations of  $HH^*(X)$  with respect to the cup-product.

The statement of this theorem is very easy for  $k = 0$ , but I do not know elementary proofs for higher values of  $k$ .

In the case of  $X$  being a compact Calabi-Yau variety one can identify the vector spaces  $HH^*(X)$  and the usual cohomology  $H^*(X)$  using a Hodge decomposition and the convolution with a volume element on  $X$ . The operator corresponding to the class  $ch_1(T_X) = c_1(T_X) = 0$  is zero, but higher operators are in general non-zero. They correspond after the identification of  $HH^*(X)$  with  $H^*(X)$  to the multiplication operators by  $ch_{2k+1}(T_X)$ .

In general, the result is that the algebra  $HH^*(X)$  has an infinite set of commuting automorphisms labeled by odd positive integers, and it is a module over a solvable quotient of the motivic Galois group. The Lie algebra of this quotient has the following basis and bracket:

$$L_0, Z_3, Z_5, Z_7, \dots, [L_0, Z_{2k+1}] = (2k+1)Z_{2k+1}, [Z_{2k+1}, Z_{2l+1}] = 0 .$$

## 5 Relations to quantum field theories

### 5.1 Local fields and $d$ -algebras

There is no satisfactory definition yet of a quantum field theory (QFT). One expects at least that a QFT on a manifold  $X$  (“the space-time”) in the Euclidean framework gives a super vector bundle of “local fields”  $\Phi$  on  $X$  (the bundle  $\Phi$  may be infinite-dimensional), and correlators

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \in \mathbb{C}, \quad n \geq 0$$

which are even polylinear maps from the tensor product of fibers of  $\Phi$  at pairwise distinct points

$$\langle \dots \rangle : \Phi_{x_1} \otimes \dots \otimes \Phi_{x_n} \longrightarrow \mathbb{C}$$

depending smoothly and  $S_n$ -equivariantly on

$$(x_1, \dots, x_n) \in \text{Conf}_n(X) = X^n \setminus \text{Diag}$$

Also one expects that there is an *operator product expansion* of fields at points converging to one point. I am not going to discuss here in details the properties of the operator products expansion.

It seems very plausible that with an appropriate definition one can prove the following

**Conjecture 6** *For any QFT on vector space  $V = \mathbb{R}^d$  with invariance under the action of the group  $G_d$  of parallel translations and dilatations (in particular, for any conformal field theory), on the tensor product*

$$\Phi_0 \otimes \wedge^*(V^*)$$

*there is a structure of a  $d$ -algebra.*

This implies that translation-invariant differential forms on  $\mathbb{R}^d$  with values in local fields form a homotopy Lie algebra  $\mathfrak{g}^*$  whose graded components are

$$\mathfrak{g}^k = \Phi_0 \otimes \wedge^{k+d-1}(V^*), \quad k \in \{-(d-1), \dots, +1\}.$$

The moduli space associated with this homotopy Lie algebra is unobstructed because  $\mathfrak{g}^2 = 0$ , and it can be interpreted as the formal deformation theory of our QFT as a translation-invariant theory, with all renormalizations and regularizations automatically included in the structure of  $L_\infty$ -algebra on  $\mathfrak{g}^*$ . Presumably, the construction of (higher) brackets on  $\mathfrak{g}^*$  is closely related with the Hopf algebra studied by Connes and Kreimer (see [CK]), where the case of free massless theory is considered.

Also, it seems that the notion of an action of  $(d+1)$ -algebra on a  $d$ -algebra is closely related with field theories on manifolds with boundaries. This subject became very popular in modern string theory after the Maldacena conjecture on boundary conformal field theories for QFT on anti-de-Sitter spaces (Lobachevsky spaces). Here I should notice that a work of Moshé and co-authors (see [FFS] and references therein) was one of the predecessors of the modern AdS picture.

## 5.2 Action of the motivic Galois group on the moduli space of QFTs

In the construction of perturbations of a given conformal field theory one needs to calculate (and often regularize) integrals of correlators over configuration spaces. In the free theory case the integrals are exactly Feynman integrals in the diagrammatic expansion.

D. Broadhurst and D. Kreimer (see [BK]) observed that all Feynman diagrams up to 7 loops in any QFT in *even dimensions* gives same numbers as appear in Drinfeld associator. It is not clear a priori why this happens. In any case one can see immediately from formulas that all constants are in fact periods.

**Conjecture 7** *The motivic Galois group  $G_{M,Betti}$  acts (in homotopy sense) on the homotopy Lie algebra  $\mathfrak{g}^*$  associated with the free massless theory in any dimension. In the case of even dimension the action factors through the quotient group  $GT$  as in 3.4. The action should be somehow related with the action on values of Feynman integrals.*

I have only one “confirmation” of this conjecture, that is the deformation quantization story. In my proof of the formality theorem, the integrals which appear in the explicit formula come from Feynman diagrams for a perturbation of a free two-dimensional quantum field theory on the Lobachevsky plane (see [CF]).

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